

ON THE FOUNDATIONS OF MATHEMATICS

1907

OVER DE GRONDSLAGEN □ DER WISKUNDE □

ACADEMISCH PROEFSCHRIFT

TERVERKRIJGINGVANDEN GRAADVANDOCTOR
IN DE WIS- EN NATUURKUNDE AAN DE UNI-
VERSITEIT VAN AMSTERDAM, OP GEZAG VAN
DEN RECTOR MAGNIFICUS Dr. J. ROTGANS,
HOOGLEERAARIN DE FACULTEIT DER GENEES-
KUNDE, IN HET OPENBAAR TE VERDEDIGEN OP
DINSDAG 19 FEBRUARI 1907 DES NAMIDDAGS
TE 3 URE IN DE AULA DER UNIVERSITEIT
DOOR LUITZEN EGBERTUS JAN BROUWER,
□ GEBOREN TE OVERSCHIE □

MAAS & VAN SUCHTELEN
AMSTERDAM—LEIPZIG. MCMVII.

I. THE CONSTRUCTION OF MATHEMATICS

Arithmetic of integers

'One, two, three, ...', we know by heart the sequence of these sounds (spoken ordinal numbers) as an endless row, that is to say, continuing for ever according to a law, known as being fixed.

Besides this sequence of sound-images we possess other sequences proceeding according to a fixed law, for instance the sequence of the written signs (written ordinal numbers) 1, 2, 3, ...

These things are intuitively clear.

Let me now stop the sequence, e.g. at 23, and let me copy beneath it the same interrupted sequence; then there exists a one-to-one correspondence between the two sequences. If I interchange two of the numbers in the upper sequence, the one-to-one correspondence persists. By such interchanges I can obtain a correspondence between a definite element of the first sequence and the element 1 of the second sequence; then a correspondence between a definite element from the remaining elements of the first sequence and the element 2 of the second sequence, etc. In other words, I can introduce an 'arbitrary order' in the elements of the first sequence, but the sequence of ordinals in the second row corresponding to it remains the same. It follows that any fixed set of signs, once counted, will produce the same 'natural number' if it is counted in a different order, that is to say, the sequence of ordinal numbers to which it is brought into a one-to-one correspondence, will be interrupted at the same number (Fundamental Theorem of Arithmetic).

By $3+4$ I mean the following: First count up to 3, then count on, but let the elements after 3 correspond one-to-one with the sequence of ordinals 1 ... 4. It follows from the fundamental theorem of arithmetic: $3+4 = 4+3$ ¹⁾. Likewise $(3+4)+5 = 3+(4+5)$.

By 9×4 I mean: Count up to 4, write 1 on another line, add 4 on the first line (the operation '+4' described above), write 2 on the second line, etc., till 9 has been written on the second line. By 9×4 I mean then the last number on the first line. By applying the fundamental theorem of arithmetic it is easy to deduce:

$$9 \times 4 = 4 \times 9; \quad (9 \times 4) \times 5 = 9 \times (4 \times 5);$$

$$9 \times (4+5) = (9 \times 4) + (9 \times 5).$$

By 4^5 I mean: First count up to 4, then write on another line the figure 1;

¹⁾ Indeed, $3+4$ leads to 1-2-3-4-5-6-7, where 4-5-6-7 corresponds one-to-one with 1-2-3-4. By permutation I obtain 4-5-6-7-1-2-3. Here 4-5-6-7 still corresponds with 1-2-3-4 and 1-2-3 with itself. Consequently the whole sequence corresponds one-to-one with that obtained by counting off $4+3$.

3

4

[[1]]

5

[[2]]

perform the operation '4 ×' described above and write the figure 2 on the second line; repeat that operation and write the figure 3, and continue until the figure 5 has been written on the second line. By 4⁵ we mean the last number written on the first line.

Negative numbers

Now we can continue the sequence of ordinal numbers to the left by 0, -1, -2, etc., define the addition of integers by counting in two directions, and from this subtraction and multiplication by a positive number; thereupon the operation -() can be introduced and we can prove that this operation commutes with multiplication. From this fact we obtain the definition and the properties of multiplication by a negative number.

Rational numbers

By a rational number we mean a pair of ordinal numbers written in the form $\frac{a}{b}$,

wherein we can always take care, by putting $\frac{a}{-b} = \frac{-a}{b}$, that the second number,

the 'denominator', is positive. We order them by putting $\frac{a}{b} \begin{smallmatrix} \geq \\ < \end{smallmatrix} \frac{c}{d}$ if $(a \times d) \begin{smallmatrix} \geq \\ < \end{smallmatrix} (b \times c)$

6 and insert them between the ordinal numbers by putting $\frac{a}{1} = a$. By $\frac{a}{b} + \frac{c}{d}$ we mean

$\frac{ad+bc}{bd}$; by $\frac{a}{b} \times \frac{c}{d}$ we mean $\frac{ac}{bd}$. The commutative, associative and distributive

properties are now easily proved; it also follows easily, if we define '-' and ':' in the well-known way from '+' and '×', that $\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$ and that $\frac{a}{b} : \frac{c}{d} = \frac{ad}{bc}$.

Irrational numbers

Next we can introduce successively the usual irrationals (first of all the expressions containing fractional exponents) by writing them as symbolic aggregates of previously introduced numbers ¹⁾ and then looking upon each of these as defining a partition of the earlier introduced numbers into two classes, of which the second follows as a whole after the first and has no first element ²⁾; the ordering relation (that is to say the condition for $\begin{smallmatrix} \geq \\ < \end{smallmatrix}$) of the newly introduced numbers between the old ones is then defined on the basis of this partition; likewise the

¹⁾ Thus, for instance, a root of an algebraic equation will be read as a symbolic aggregate of its coefficients, completed by a number which orders the roots e.g. first after their absolute value and, in the case of equal absolute values, after the argument.

²⁾ while in some cases the first class may have one of the previously introduced elements as its last element (like 2 in the case of 4 $\frac{1}{2}$).

operations with the new numbers, which in their turn may give rise to new numbers, and finally the numbers introduced earlier are brought into one-to-one correspondence with part of the new symbols, viz. with those which in the old numbers determined a lower class with a last element. The symbolic aggregates which have been introduced can contain any finite number of previously introduced numbers. This implies that at each stage of development of the theory the set of the numbers known remains denumerable ¹⁾. Indeed, a denumerable set of denumerable sets is denumerable, as Cantor established by a simple argument (Journ. Math. 84 (1878), p. 243 [Ges. Abh. p. 120]).

Further the set of numbers which have been introduced at any stage in the development of the theory, has the property of being *everywhere dense in itself*, this means that between any two of them there are other elements ²⁾. Consequently, according to Cantor (Math. Annalen 46 (1895), p. 481–512 [Ges. Abh. p. 282–351]) it has the order type η , that is, it can be mapped on the system of the rationals by an orderpreserving mapping.

It would not be difficult to introduce the new numbers in such a way that the order type η would not be obtained – we shall see examples of this when we shall build up geometry – but it is done in the way indicated above for reasons of efficiency, connected with the creation of the *measurable continuum*, which we shall now take into consideration.

The continuum

In the following chapters we shall go further into the basic intuition of mathematics (and of every intellectual activity) as the substratum, divested of all quality, of any perception of change, a unity of continuity and discreteness, a possibility of thinking together several entities, connected by a ‘between’, which is never exhausted by the insertion of new entities. Since continuity and discreteness occur as inseparable complements, both having equal rights and being equally clear, it is impossible to avoid one of them as a primitive entity, trying to construe it from the other one, the latter being put forward as self-sufficient; in fact it is impossible to consider it as self-sufficient. Having recognised that the intuition of continuity, of ‘fluidity’, is as primitive as that of several things conceived as forming together a unit, the latter being at the basis of every mathematical construction, we are able to state properties of the continuum as a ‘matrix of points to be thought of as a whole’.

First of all, there is neither a first nor a last point; it is easy to construct on the continuum a sequence of points having the order type of the positive and negative

¹⁾ that is to say, it can be brought into uniform correspondence with the sequence of the real numbers.

²⁾ In particular between any two of them there is an infinity of rational numbers, a property which we express by saying that the system of the rationals is *relatively dense* in the set of the numbers introduced at that stage.

whole numbers; if we add a point in every interval, then again in each of the intervals so obtained, and so on, we obtain the order type η on the continuum, which in this way comes to correspond with the system of finite dual fractions ¹⁾, but we might read it as well as one of the dense systems of numbers introduced above; now we see immediately that on the continuum there are still points which have not been reached by a finite number of the operations described above, each being the insertion of a point in every interval ²⁾; for we can, having fixed a point P, take care to avoid it in the construction of the scale. We can even arrange the construction in such a way that the approximation of the point by an infinite dual fraction is given by an arbitrarily given law of progression, though the continuum with the scale constructed in this way differs in no respect from a continuum on which the scale is constructed in complete freedom. Conversely we see that for any scale which has been constructed on the continuum, there exists a point corresponding to any conceivable law of progression.

However, we can never consider the approximating sequence of a *given definite* point as being *completed*, so we must consider it as partly unknown.

From the fact that every conceivable approximating sequence occurs it can be deduced, following Cantor (Jber. Deutsch. Math.-Ver. 1 (1890–91), p. 77 [Ges. Abh. p. 278–280]); compare Schoenflies, Die Entwicklung der Lehre von den Punktmannigfaltigkeiten I, Jber. Deutsch. Math.-Ver. 8, part 2, Leipzig 1909, p. 20) that it is impossible to enumerate all the points of the continuum, i.e. there are points lying outside any denumerable set of points (whereas we have seen that the system of *constructed* numbers, which likewise can be approximated by a finite or infinite dual fraction, remains denumerable at every stage of the theory).

The measurable continuum

If we construct the dual scale arbitrarily, then it is not certain that it will be everywhere dense, i.e. that it will penetrate into every segment of the continuum. But we agree to contract every segment not penetrated by the scale into one point, in other words, we consider two points as different only when their approximating dual fractions differ after a finite number of digits. Finally, if we take some point on the constructed scale as the zero point, then the scale has made the continuum into a *measurable continuum*. From the measurability we conclude that every denumerably infinite set of points, lying in the segment determined by two points as its endpoints, has at least one limit point, i.e. at least one point such that on at least one of its sides in every segment contiguous to it there are other points of

¹⁾ i.e. the fractions written in the dual system, in which before and after the point no other digits than 0 and 1 occur.

²⁾ We may call this operation the 'bisection' of the intervals.

the set ¹⁾. (For otherwise there would be a shortest distance between points and this could be held only a finite number of times in the finite segment.)

[[5]]

12

The translation

We proceed to consider point transformations on the measurable continuum, and we start with the *translation*, which we denote by $+a$, a being the point into which the zero point is transformed. This notion is immediately clear in virtue of the scale, and we see that these operations form a *group* (this is a consequence of the associative property of addition) and also that it is commutative (this means that $a+b = b+a$, i.e. the operation $+b$ applied to a gives the same result as the operation $+a$ applied to b). ²⁾

[[8]]

The following properties hold for this group:

1°. It is a *one-parameter continuous* group, i.e. its transformations can be arranged in a linear continuum in such a way that, if a point moves continuously in this image-continuum, then the corresponding transformations bring about simultaneous continuous movements for the points of the continuum to which they are applied.

2°. It is *uniform*, that is: every transformation maps any two different points to different points.

13

3°. It is *closed*, that is: if $A_1, A_2, A_3 \dots$ is a denumerably infinite sequence of points having A as a limit point on the measurable continuum, and if likewise $B_1, B_2, B_3 \dots$ is a denumerably infinite sequence of points having B as a limit point, while there is a transformation in the group which maps the pair of points $A_1 B_1$ to $A_2 B_2$, likewise a transformation which maps $A_1 B_1$ to $A_3 B_3$, and so on, then there is a transformation which maps $A_1 B_1$ to AB .

Definition of addition on the continuum in terms of a group

Let there now be given an arbitrary group of transformations on the measurable continuum, having the 3 properties mentioned above, thus being one-parameter

¹⁾ By constructing on an open interval between two points of the continuum a sequence of points having the order type of the positive and negative whole numbers, and approaching indefinitely, according to the measure of the dual scale, the endpoints of the interval, and by constructing in this sequence a new dual scale, penetrating into every subinterval of the given interval, we show that this interval, considered as a matrix of points, is equivalent to the whole continuum; both of them form a so-called *open continuum*.

[[6]]

Starting from here we can construct as follows the *closed continuum*. $\alpha \gamma^P \delta \beta$ Any point P on an open continua $\alpha\beta$ has to its left and to its right two new open continua $\alpha\gamma$ and $\delta\beta$. Conversely, out of two open continua $\alpha\gamma$ and $\delta\beta$ we construct a new open continuum $\alpha\beta$ coupling them together by means of a single point P . But now we can repeat this operation analogously by coupling together α and β , inserting a single point; thus we obtain a closed continuum.

[[7]]

²⁾ For points of the constructed scale commutativity is a consequence of the fact that addition of rational numbers is commutative; for points a and b not on the scale $a+b$ and $b+a$ give the same sequence of approximating points of the scale, and their equality follows then from the measurability of the continuum.

[[19]]

continuous, uniform and closed. Then we can derive the following further properties.

4°. The order of the points must remain unchanged by every transformation, for otherwise two points whose order is reversed would have met on their continuous paths and uniformity would be disturbed.

5°. Two different points cannot be brought arbitrarily near to each other by transformations in the group. For then, because the group is closed, they could be mapped together on a common limitpoint, and this contradicts uniformity.

6°. The image of a limitpoint of a sequence of points is a limitpoint of the image of the sequence. For if we select from the first sequence a subsequence p in which each point is to the right (resp. to the left) of the preceding point, and which approximates in this way the limitpoint P , then this sequence is mapped to a denumerably infinite sequence q , in which each point is to the right (to the left) of the preceding one. Let Q be the image of P , then there is no point between Q and all the points q , because there is no point between P and all the points p , so Q is limitpoint of q .

Now let us choose some point as the zero point. We start with a transformation in the group which maps the point 0 to the point a and denote this transformation by '+ a '. Further we denote the image of a by $2a$, the image of $2a$ by $3a$, etc. The transformation now determines a uniform correspondence between the segments constructed in this way.

Somewhere between the point 0 and the point a there must be a point b such that the transformation that carries 0 to b , carries b to a , so that the operation '+ b ', when applied twice in succession, is equivalent with the operation '+ a '. We put $b = \frac{1}{2}a$ and the corresponding points in the further segments between na and $(n+1)a$ analogously $= \frac{3}{2}a, \frac{5}{2}a$, etc. The continuum is now divided into segments $= b = \frac{1}{2}a$, and the points of all these segments are in uniform correspondence.

Continuing in this way, we construct from the operation '+ c ' which, when applied twice in succession, is equivalent with '+ b ', a division of the continuum into segments $= c = \frac{1}{2}b = \frac{1}{4}a$, and finally we obtain a complete dense dual scale on the continuum, which has the following properties:

a. It has neither to the left nor to the right a limiting point; for otherwise the points $a, 2a, 3a$, etc. would have a limitpoint, and the points a and $2a$ could be brought arbitrarily near to each other in the neighbourhood of that limitpoint, which contradicts the property mentioned under 5°.

b. It is dense on the measurable continuum, that is to say it penetrates every segment of the measurable continuum. For, to begin with, it is clear that the scale possesses arbitrarily small segments; suppose now that there were on the measurable continuum a segment in which the scale did not penetrate, then we could choose in that segment two points A and B , and for any segment of the scale there would be transformations mapping those two points to points of that

segment; as there exist arbitrarily small segments of the scale, A and B could be brought arbitrarily near to each other, again in contradiction to property 5°.

It follows from the properties a) and b) that the scale, belonging to the group, has made the continuum measurable in a new way, leading to the property:

16

7°. The group is commutative.

Finally we remark that by repetition of an arbitrary continuously applied transformation from the group *any* transformation from the group will be passed; we express this fact by

8°. The group parameter is measurable.

Thus we have seen that a continuum, once it is known to be measurable, can be measured in an unlimited number of ways, for to every one-parameter continuous, uniform, closed group there corresponds a measure; conversely to every measure there corresponds a one-parameter continuous, uniform, closed group.

If we leave out condition 3° for the group, then also property 5° lapses. It is still possible to construct a dense scale corresponding to the group, but the properties a) and b) can no longer be proved by the method indicated above, nor can the properties 7° and 8° for the group. However, let us consider more closely such a scale, dense in itself, but not satisfying property b).

We see then that, while in the corresponding group a point of the continuum moves inside a free interval, the limit points of the scale remain invariant. Thus we have transformations which, even repeated indefinitely, leave some points invariant, which do move by other transformations; repetition of continuous transformations of the first kind never yields a transformation on the other side of a transformation of the second kind. We express this phenomenon by calling the group *non-measurable* or *non-Archimedean*. But further we find in this group transformations which map the whole free interval to a single limiting point of the scale, thus the group is not uniform. It follows that property b) of the scale is a consequence of the properties 1° and 2° of the group.

17

Things are different for property a). For the group of translations corresponding to a dense scale on the continuum, which is bounded to the left or to the right or on both sides by a limiting point, is one-parameter continuous in the domain of the scale. Only we must remark that the points in the domain can overstep the boundary points by no transformation of the group and that the boundary points themselves are invariant. We see further that properties 3° (closure) and 5° hold for the domain of the scale, provided we exclude explicitly the boundary points, for there it does happen that two different points become arbitrarily near to each other, while there is no transformation of the group which transforms both into the boundary point.

The points outside the domain of the scale can remain invariant by the group, or can be transformed inside domains of other scales, separated by invariant boundary points. Thus the group determines on the measurable continuum a finite or infinite sequence of contiguous intervals, each of which is either [point-

18

wise] invariant or is transformed according to a one-parameter continuous group, closed outside the boundary points.

The invariant boundary points of the segments are called the *fixed points* of the group.

The group is commutative in every segment between two fixed points, and the group parameter is measurable. (Both properties follow from the density of the scales.)

Summarizing, we have stipulated that certain groups on the continuum are

1° one-parameter continuous,

2° uniform,

and we have deduced for every such group the further properties:

[9] 3° it divides the continuum in finite intervals of which the boundary points, the fixed points of the group, are invariant, and each of which is either transformed by a one-parameter uniform group without fixed points, or is [pointwise] invariant.

19 4° The order of the points of the continuum remains unchanged by the transformations of the group.

5° The group is closed outside its fixed points.

6° Outside the fixed points two different points cannot be brought arbitrarily near to each other.

7° A limitpoint of a sequence of points is transformed by every transformation of the group into the limitpoint of the transformed sequence.

8° On an interval between two fixed points, which is not [pointwise] invariant, the group defines a dense scale [after the choice of the points 0 and 1].

9° The group is commutative.

10° The group parameter is measurable.

Thus, having previously characterized the *group of additions on the open continuum* as a one-parameter continuous uniform closed group on the measurable continuum, we can now give a more ample characterization as a one-parameter uniform group on the measurable continuum, restricted to the domain between two of its fixed points. – Now one can consider these fixed points, being the end-points of the domain of the group, as coinciding points, thus obtaining the *operation of addition on the closed continuum*; the locking-point, arising from the coinciding boundary points, is called the *point at infinity* of the group. (In order to map the transformations of the group on the points of its domain, we have introduced another special point, the *zero point*, which however does not, like the point at infinity, play a special part for the group itself.)

20 With respect to a given scale an arbitrary one-parameter continuous uniform group can be very complicated, but on its own scale, as we have constructed it from the group, it is represented on every interval, bounded by two successive fixed points, by the group of additions. Let us consider, for instance, the group

of multiplications on a given scale ¹⁾; it has the point 0 as a fixed point, so it has two separate domains, namely between $-\infty$ and 0 and between 0 and $+\infty$; its own scale coincides on the latter domain with the scale of $\log x$, on the former with that of $\log(-x)$; on its own scale the group is in either domain the group of additions.

Definition of multiplication on the continuum in terms of a group

We now intend to find the most general set of two one-parameter continuous uniform groups that can be combined to a two-parameter continuous group ²⁾, and to represent this group as simply as possible on a special scale. First of all we remark that, no matter how the fixed points of the component one-parameter groups are distributed, we can divide the continuum in segments, each bounded by two fixed points of one of the groups, and containing at most one fixed point of the other group. We investigate the construction of the two-parameter group in each of these segments. The most general two-parameter group consists of a juxtaposition of such segments, each having a two-parameter group with the structure described below. We choose the scale on the segment to be that of the one-parameter group which has the endpoints of the segment as its fixed points; with respect to this scale our group is the complete group of additions, and its domain with respect to an arbitrarily chosen unity of the scale extends from $-\infty$ to $+\infty$. Let us take this domain as the X -axis in a Cartesian coordinate system, and assign to each point of this axis its increment by some transformation of the second group as a Y -coordinate; then the transformations of the second group will be represented by a system of curves which, apart from a possible common intersection with the X -axis (the possible fixed point of the second group) are disjoint. We write the equations of these curves as:

$$y = f_{\alpha}(x), \quad \text{[10]}$$

and our condition amounts to stipulating that an arbitrary sequence of transformations, each belonging to one of the two groups, can be replaced by one transformation of the second group, followed by one transformation of the first group, so that it can be represented by

$$x' = x + f_{\alpha}(x) + \beta.$$

We choose as the origin some point on the X -axis, and consider the case when the group under consideration has no fixed point, so that the curves $y = f_{\alpha}(x)$ are disjoint. If one chooses the group-parameter in such a way that the curve $y = f_{\alpha}(x)$ intersects the Y -axis in the point $(0, \alpha)$, the curve $y = f_{\alpha}(x) - \alpha$ passes through the origin. Now we know that the result of applying successively two transformations giving respectively the increments $f_{\gamma}(x) - \gamma$ and $f_{\delta}(x) - \delta$ is a

¹⁾ i.e. the group of operations of multiplication by a positive number.

²⁾ i.e. a group whose transformations are determined by two continuous parameters.

transformation giving the increment $f'_\zeta(x) + \sigma$; but this must be 0 for $x = 0$, so it can be nothing else but $f'_\zeta(x) - \zeta$.

In other words, the transformations $x' = x + f'_\alpha(x) - \alpha$ form a one-parameter continuous group which, as well as that represented by $x' = x + f'_\alpha(x)$, is uniform, and which can be combined with $x' = x + \alpha$ to the same two-parameter group as $x' = x + f'_\alpha(x)$. But now the group $x' = x + f'_\alpha(x) - \alpha$ has a fixed point. Thus the case when the second group has no fixed point between two consecutive fixed points of the first group, has been reduced to the case when it possesses one such point, and we need only consider this latter case.

23 We choose the origin as the fixed point on the X -axis; now the increment functions $f'_\alpha(x)$ intersect in 0 and have no other point in common. Our object is to prove that these functions are differentiable.

Let us denote by ${}_a\varphi_\Delta(x)$ the increment of the function $f'_\alpha(x)$ between the abscissae x and $x + \Delta$. As $f'_\alpha(x)$ is continuous, so too ${}_a\varphi_\Delta(x)$. Let us suppose that ${}_a\varphi_\Delta(x)$ assumes equal values for two different values of x , say x_1 and $x_1 + p$.

In that case the system consisting of the curves $y = f'_\alpha(x) + \beta$ and $y = f'_\alpha(x + p)$ would contain two curves which intersect for $x = x_1$ and for $x = x_1 + \Delta$, so the system, consisting of the curves $y = f'_\alpha(x + x_1) + \beta$ and $y = f'_\alpha(x + x_1 + p)$, would contain two curves which intersect for $x = 0$ and for $x = \Delta$. But this system is contained in the system of curves $y = f'_\alpha(x) + \beta$, and in this system two curves which intersect for $x = 0$, cannot intersect again – unless they are the same curve. This would mean for the original curve $y = f'_\alpha(x)$ that its path from $x = x_1$ on would be homothetic to its path from $x = x_1 + p$ on, in other words it would have a periodically homothetic path with period p . Then we have ${}_a\varphi_p(x_1 + k) = {}_a\varphi_p(x_1)$ for any value of k , and this results in a periodically homothetic path with period k in the same way as ${}_a\varphi_\Delta(x_1 + p) = {}_a\varphi_\Delta(x_1)$ entailed a periodically homothetic path with period p . But if $y = f'_\alpha(x)$ has a periodically homothetic path for any period whatever, then it is necessarily a straight line, and certainly a differentiable curve.

24 We are now sure that, if the curve $y = f'_\alpha(x)$ is not a straight line, then ${}_a\varphi_\Delta(x)$ cannot assume twice the same value. So for given α and Δ this function must either always increase or always decrease, and it is immediately clear that, for a given α , it either increases for every Δ or it decreases for every Δ . (For if it increases for Δ , then also for $\frac{1}{2}\Delta$, $\frac{1}{4}\Delta$, etc., and also for 2Δ , 3Δ , etc.) Let us denote the difference quotients of $f'_\alpha(x)$ between 0 and Δ , Δ and 2Δ , 2Δ and 3Δ etc. respectively by ${}_1Z$, ${}_2Z$, ${}_3Z$, etc.; then those between 0 and $\frac{1}{2}\Delta$, $\frac{1}{2}\Delta$ and Δ , Δ and $1\frac{1}{2}\Delta$ etc. by ${}_1Z_0$, ${}_1Z_1$, ${}_2Z_0$, etc.; then those between 0 and $\frac{1}{4}\Delta$, $\frac{1}{4}\Delta$ and $\frac{1}{2}\Delta$, $\frac{1}{2}\Delta$ and $\frac{3}{4}\Delta$, etc. by ${}_1Z_{00}$, ${}_1Z_{01}$, ${}_1Z_{10}$, ${}_1Z_{11}$, ${}_2Z_{00}$, etc.; in this way we obtain an arbitrary number of indices behind the Z , and we know that if ${}_aZ_p$ and ${}_aZ_q$ have the same number of indices, and if the endpoint of the interval of ${}_aZ_p$ is the starting point of that of ${}_aZ_q$, then ${}_aZ_q$, ${}_aZ_{q0}$, ${}_aZ_{q00}$ etc. will decrease, but remain greater than ${}_aZ_p$; likewise ${}_aZ_p$, ${}_aZ_{p1}$, ${}_aZ_{p11}$, ${}_aZ_{p111}$ etc. increase, but remain less

than ${}_aZ_q$. Therefore the first sequence has a lower limit to which it tends and the latter sequence has an upper limit. That establishes the existence of a derivative at every point for $y = f_a(x)$, but it has not been established that the derivatives on the left and on the right cannot be different.

25

However this is not necessary for our purpose, for the sheer existence of the derivatives allows us, in the case of our problem, namely the investigation of the most general two-parameter continuous uniform group, to apply the fundamental formula of LIE ¹⁾ which for a two-parameter differentiable group takes the form:

$$\varphi_1 d\varphi_2 - \varphi_2 d\varphi_1 = c_1 \varphi_1 + c_2 \varphi_2,$$

where φ_1 and φ_2 denote the increments by very small transformations from the two generating one-parameter groups. To begin with, let us assume that $c_2 = 0$, so that we have

$$(1) \quad \varphi_1 d\varphi_2 - \varphi_2 d\varphi_1 = c_1 \varphi_1.$$

If we determine the points of the continuum between two successive fixed points of the group φ_1 by a coordinate x , measured on the scale of φ_1 , then we can write:

$$\varphi_1 = \varepsilon_1,$$

and for φ_2 we obtain the differential equation:

$$\begin{aligned} \frac{d\varphi_2}{dx} &= \varepsilon_2 \\ \varphi_2 &= \varepsilon_2(x+h), \end{aligned}$$

so that we find as the finite group with c as group parameter:

26

$$(x' + h) = c(x + h),$$

and the combined two-parameter group becomes:

$$x' = c_1 x + c_2.$$

In this way we have completed the one-parameter group to a two-parameter one by reading it as the group of additions on a scale, and afterwards taking for the two-parameter group that of addition and multiplication, combined on that scale. And we obtained for the second one-parameter group, to be combined with the first one, the group of multiplications on that scale with respect to an arbitrary zero point.

Next suppose that c_2 is not 0, then we can put $c_1 \varphi_1 + c_2 \varphi_2 = \varphi_3$, and between φ_1 and φ_3 we find the relation:

$$(2) \quad \varphi_1 d\varphi_3 - \varphi_3 d\varphi_1 = c_2 \varphi_3$$

¹⁾ cf. Math. Ann. 8 (1875) p. 103; Göttinger Nachr. 1874, p. 529-542; Theorie der Transformationsgruppen I, p. 150.

or, working again on the scale of φ_1 ,

$$\frac{d\varphi_3}{dx} = k\varphi_3; \quad \varphi_3 = \varepsilon e^{kx}; \quad \varphi_2 = \varepsilon_1 e^{kx} + \varepsilon_2.$$

Thus by the transformation φ_2 , x suffers an increment of

$$dx = \varepsilon_1 e^{kx} + \varepsilon_2,$$

and e^{-kx} suffers an increment of:

$$d(e^{-kx}) = \varepsilon_3 + \varepsilon_4 e^{-kx},$$

27 so that, if we put $e^{-kx} = \xi$, and if we work in the scale of the translation group of ξ :

$$\text{either } \varphi_2(\xi) = \varepsilon_4(\xi + h) \quad \text{or} \quad \varphi_2(\xi) = \varepsilon_3,$$

which gives as the finite group:

$$\text{either } \xi' + h = c(\xi + h) \quad \text{or} \quad \xi' = \xi + c.$$

The original one-parameter group becomes in the scale of ξ :

$$\xi' = c\xi,$$

and the combined two-parameter group becomes:

$$\xi' = c_1\xi + c_2.$$

We might also have gathered directly from equation (2) that, working on the scale of φ_3 , we find for φ_1 a multiplication-group, consequently for φ_2 one of the groups:

$$(x' + h) = c(x + h).$$

28 In this way we have completed the one-parameter group to a two-parameter one by reading it as 'multiplication-group between the points 0 and ∞ ' of a scale (i.e. by taking the scale of e^{-kx} if x is the scale of the group as a group of additions¹⁾) and afterwards taking as the two-parameter group that of addition and multiplication combined on that scale. The second component one-parameter group is either the group of additions on that scale, or the multiplication-group with a zero-point (i.e. fixed point) different from the zero-point of the first group. The completion can take place in different ways, depending upon the value of k ; to every completion there corresponds a scale which plays the part of the group of additions in the two-parameter group, and by the choice of the scale of the

¹⁾ Properly speaking, we obtain the 'multiplication-group between 0 and ∞ ' only for negative k ; if k is positive we find $\xi = \infty$ for $x = -\infty$; as in this case, in order to have an increasing scale for x , we prefer to use the scale of $-e^{-kx}$ instead of that of e^{-kx} , we obtain our original group as 'multiplication-group between $-\infty$ and 0'.

group of additions it is determined how the group will be completed. Also the chosen group of additions is the only one with respect to which *both* the one-parameter groups which occur as the components of the two-parameter group, and consequently also the two-parameter group itself, play the part of *similarity transformations*¹⁾; in particular the two-parameter group is with respect to that scale the group of all the similarity transformations, and by the two-parameter group as a group of similarity transformations the scale on the whole domain in question is determined.

Further we have remarked that in this case one of the bounding fixed points of the original one-parameter group remains a fixed point for the two-parameter group, but the other does not; the second component one-parameter group has within the domain of the original group one or no fixed point.

Let us now consider again the two one-parameter groups which could be combined into one two-parameter group, extended over the whole continuum, as we originally imagined them; let us denote by P_α the fixed points for *both* groups, which consequently are fixed points as well for the two-parameter group which is combined from them, by Q_α those of the first and by R_α those of the second group. Any two points Q, and likewise any two points R, will be separated by at least one point P; in other words, between two points P there can be at most one point Q and one point R.

Next we consider the domain between two consecutive points P, P_1 and P_2 say; and we imagine between them a point Q and a point R.

$$P_1 \quad Q \quad R \quad P_2$$

If then we apply the last results on the domain between two consecutive fixed points of one of the component groups in turn to the domain QP_2 and to the domain P_1R , we find:

Between Q and P_2 there is a scale, bounded in Q ²⁾ but unbounded towards P_2 ; the first component one-parameter group consists of the similarity transformations of this scale with Q as their fixed point, the second consists of the similarity transformations with R as their fixed point; the resulting two-parameter group consists of *all* similarity transformations. The part of the scale between Q and R is the only scale between Q and R with respect to which the two-parameter group can play the part of a similarity group.

Also between P_1 and R there is a scale, bounded in R, but unbounded towards P_1 ; the first component one-parameter group consists of the similarity transformations on that scale with Q as their fixed point; the second consists of the similarity transformations with R as their fixed point; the resulting two-parameter group consists of *all* similarity transformations. The part of the scale between Q

¹⁾ By this we understand: both with invariant proportions and the same orientation.

²⁾ For concision we call a scale bounded in a point when that point can be reached according to the translation group built on that scale.

and R is the only scale between Q and R , with respect to which the two-parameter group plays the part of a similarity group. Thus the part of *this* scale between Q and R is identical with the part between Q and R of the scale considered above, so that we can say now:

Between P_1 and P_2 there exists a scale, unbounded in both directions; the first component one-parameter group consists of the similarity transformations with Q as their fixed point (group of multiplications with Q as zero); the second consists of the similarity transformations with R as their fixed point (group of multiplications with R as zero). The resulting two-parameter group is the similarity group (or combined group of multiplication and addition).

31 Let us now suppose that between P_1 and P_2 there is only a point R , but no point Q ; this happens only in our first way of completing a one-parameter group to a two-parameter one, namely where we consider the first one-parameter group as a group of additions between two consecutive of its fixed points, and where we choose as the second one-parameter group the group of multiplications with an arbitrary zero; again the two-parameter group becomes the similarity group on the scale constructed between P_1 and P_2 .

Finally it is impossible that between P_1 and P_2 there is neither a point Q , nor a point R , for we have seen that between consecutive fixed points of a one-parameter group we cannot add a further one-parameter group to yield a two-parameter group conserving the two fixed points without having another fixed point between them.

We have now deduced the following results about two one-parameter continuous uniform groups, on the measurable continuum, which can be combined to a two-parameter group:

32 On the measurable continuum there is a finite or an infinite sequence of contiguous finite segments, the boundary points of which are invariant for the two-parameter group and are called *fixed points* of the two-parameter group. On each of these segments there can be constructed a dense scale, unbounded on both sides, so that each component one-parameter group plays one of the following parts on each segment:

- either it leaves the segment $\llbracket \text{pointwise} \rrbracket$ invariant,
- or it plays there the part of a group of additions,
- or it plays there the part of a multiplication group with an arbitrary zero point.

Generally speaking, the same one-parameter group will play different parts in different segments.

The resulting two-parameter group will, in general, play in each segment the part of a similarity group, but there may be certain segments where it is reduced to a one-parameter group (which then can be read as a group of additions) or which it leaves $\llbracket \text{pointwise} \rrbracket$ invariant.

On a given scale an arbitrary two-parameter continuous uniform group can be very complicated, but on its own scale, which can always be constructed from

the group, it is represented on each segment, bounded by consecutive fixed points, by the group of additions and multiplications.

The object of the foregoing development was to define the operations of addition and multiplication on the complete measurable continuum as follows (meaning by multiplication only that by a positive factor):

Operation of addition on the complete measurable continuum:

One-parameter continuous uniform group on the measurable continuum between two consecutive fixed points.

Operation of multiplication on the complete measurable continuum:

One-parameter continuous uniform group between the same fixed points as the preceding one, which can be combined with it to a two-parameter group.

We can also start by defining the combination of addition and multiplication as the two-parameter continuous uniform group on the measurable continuum between two consecutive fixed points, then addition either as the only one-parameter subgroup without fixed point in that domain, or as the only invariant subgroup, and multiplication as any other one-parameter subgroup. ¹⁾

Redundant part of the distributive property

This group-theoretical definition of the arithmetical operations on the measurable continuum shows that, in the axiomatic definition of these operations, after the associative and commutative properties of addition and of multiplication have been given, the complete distributive property is no longer necessary in order to define completely the operations. For if we denote the operation of addition by f_α , that of multiplication by φ_α , then the distributive property affirms that

$$\varphi_\alpha\{f_\beta\} = f_{\varphi_\alpha(\beta)}\{\varphi_\alpha\},$$

while we have shown that the following condition is sufficient:

$$\varphi_\alpha\{f_\beta\} = f_\gamma\{\varphi_\delta\}.$$

Inversion of sign and multiplication by a negative factor

If we add now the transformation $x' = -x$, we see that it can be associated with the group of additions; the result is still a one-parameter uniform group, but with a discontinuity in the group-parameter. Likewise it can be associated with the group of multiplications, giving a group of multiplications by a positive or negative factor, which is also a uniform one-parameter group with a discontinuity; and finally also the combined two-parameter group can be doubled by it over a discontinuity. ²⁾

¹⁾ On the *closed continuum* we can make both bounding fixed points coincide; thus the group of addition and multiplication can also be understood as the general two-parameter continuous uniform group on the closed measurable continuum with only one fixed point.

²⁾ This means here for a two-parameter group: a line of discontinuity in the parameter plane.

Let us now add the transformation $x' = \frac{1}{x}$; it can be associated with the complete group of multiplications, giving a uniform one-parameter group with a new discontinuity; and with the complete two-parameter similarity group, giving a uniform three-parameter group, the so-called *projective group*, of the transformations $x' = \frac{ax+b}{cx+d}$.¹⁾

LIE has proved for differentiable transformations that there exists only one construction for a three-parameter continuous group on the measurable continuum, namely the projective group. Above it has been proved independently of the differentiability that there exists only one construction for a one-parameter, respectively a two-parameter continuous uniform group; it is left here as a problem to relax the condition of differentiability for the three-parameter group as well.²⁾ [Brouwer solved this problem in Math. Annalen 67 (1909), p. 246–267 (Vol. 2, 1909 C); Math. Annalen 69 (1910), p. 181–203 (Vol. 2, 1910 H). See (1917), remark 1 [p. 145].]

Projective geometry

In order to build up projective geometry we take $n+1$ open continua, each with a translation group and a zero (then on each of them there is also defined a multiplication group). We can map all or part of these $n+1$ translation groups with their zeros on one another; every continuum participating in the mapping can do that in two directions. The agglomeration of these groups, as they take part in the mapping, is called a *point* P. In the agglomeration the same group can occur more than once; then we can replace these different mappings by a new mapping, in which for the corresponding points in the components their sum is substituted. However, the groups of different continua cannot be united in the sum.

The groups of an arbitrarily chosen point can be transformed, each by a multiplication-operator, in those of another arbitrary point, and for these multiplications only the ratio of the factors is determined. If we take the first point as a basis point, then any other point is determined by these ratios, its *coordinates*. We see here that a point cannot be given directly by coordinates, i.e. numbers from a continuum with a scale, but only with respect to another point.

As the different groups of one and the same point depend together upon one

¹⁾ This three-parameter group also has a surface of discontinuity in the parameter-space, separating the transformations with one orientation from those with the opposite orientation.

²⁾ If we think of the projective group on the closed continuum and if we keep some point fixed, then it is clear that we are left with a two-parameter group which has that point as its only fixed point; consequently this group can be read as a *group of arithmetical operations*; compare F. SCHUR's 'Rechnen mit projektiven Strecken' (Math. Annalen 55 (1902), p. 265–292) and also HILBERT's 'Endenrechnung' (Math. Annalen 57 (1903), p. 137–151 [Anhang III in 'Grundlagen der Geometrie']).

parameter, we can also consider the point as a single translation group with a zero point. And we can map upon one another the translation groups of two, three, ..., $n+1$ points, and take the sum of the groups occurring in the mapping; this sum will always give one of the points defined above. The points which are derived in this way by the mapping upon one another of the points A_1 and A_2 , are said to *lie on the straight line* ($A_1 A_2$); those derived from the mapping upon one another of $p+1$ points $A_1, \dots, A_{p+1}$ are said to *lie in the flat p space* ($A_1 A_2 \dots A_{p+1}$). We see that, if A_1 lies in ($A_2 A_3 \dots A_q$), then A_2 lies in ($A_1 A_3 \dots A_q$), etc., and that the flat spaces are represented in the coordinates by systems of linear equations. Further it can be proved that, if $A'_1, \dots, A'_{p+1}$ are points of the flat p space ($A_1 A_2 \dots A_{p+1}$), not lying in a $p-1$ space, then every point of the p space can be derived from the points A' as well as from the points A ; likewise, that a p space and a q space in a $p+q$ space have one point or a straight line or a 2 space etc. in common.

37

Let us consider a p space in which we take as basis points $A, B_1, \dots, B_p$, and let us call the corresponding coordinates $\alpha, \beta_1, \dots, \beta_p$. Let $F_1, \dots, F_p$ be points on (AB_1), (AB_2), ..., (AB_p); let G_1 and G_2 be points in ($B_1 \dots B_p$), and C_1 and C_2 the points of intersection of AG_1 and AG_2 with ($F_1 \dots F_p$). Let us suppose that we obtain G_2 from G_1 by multiplying the coordinates $\beta_1, \dots, \beta_p$ of G_1 by $h_1, \dots, h_p$. Then the same holds for C_2 and C_1 , for these points have the same coordinates $\beta_1, \dots, \beta_p$ as G_2 and G_1 . But now let us determine C_2 and C_1 with respect to $A, F_1, \dots, F_p$, and suppose that from C_1 to C_2 we must multiply the coordinates $\varphi_1, \dots, \varphi_p$ by factors $h'_1, \dots, h'_p$. Then we have:

$$\begin{aligned} C_1 &= \varphi_1 \kappa_1 (B_1 + a_1 A) + \dots + \varphi_p \kappa_p (B_p + a_p A) = \\ &= \varphi_1 \kappa_1 B_1 + \dots + \varphi_p \kappa_p B_p + \sum \varphi_i \kappa_i a_i A. \end{aligned}$$

38

$$C_2 = h'_1 \varphi_1 \kappa_1 B_1 + \dots + h'_p \varphi_p \kappa_p B_p + \sum h'_i \varphi_i \kappa_i a_i A.$$

But the coefficients of $A, B_1, \dots, B_p$ in these formulae are the coordinates $\alpha, \beta_1, \dots, \beta_p$ in the original system; thus we see that the h are the same as the h' . Consequently the relative coordinates of points in one and the same p space are *projective*, in particular this holds for the cross ratio of 4 collinear points.

Once these principles have been given, the further development of projective geometry, in the first place the construction of the quadratic loci, provides no more difficulties.

Cartesian geometry

In order to build up Cartesian geometry, we take n open continua and we understand by a point of "R the combination of n points of these different continua. Such a point of "R can be defined by n coordinates, to wit numbers taken from one continuum, provided with a scale, if we choose on each of the n given continua a scale, unbounded in both directions (thus a translation group), and a unit segment.

Euclidean geometry

The Cartesian geometry further becomes a *Euclidean geometry* if, in addition, we define the *distance* between two points by $\sqrt{\sum (x'_i - x_i)^2}$.

Then one observes that the transformation group

$$X_1 = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n + \beta_1$$

etc.

39 with $\sum_q \alpha_q^2 = 1$ and $\sum_q \alpha_q \alpha_q = 0$ leaves the distance between any two points invariant, and that conversely the group is determined by the condition that $\sqrt{\sum (x'' - x')^2}$ is invariant. The group is called the *Euclidean congruence group* of $\mathbb{R}$.

The similarity group

If one writes in the left members of the equations cX_1, cX_2 , etc. instead of X_1, X_2 , etc., then one obtains the *group of similarity transformations*, by which all the distances are multiplied by the same factor.

The group of complex operations

In $\mathbb{R}^2$ the subgroup of the similarity transformations with positive determinant becomes:

$$\begin{aligned} X_1 &= ax_1 - bx_2 + d_1 \\ X_2 &= bx_1 + ax_2 + d_2. \end{aligned}$$

We call it the *group of complex operations*, after having understood by a complex number $x_1 + ix_2$ a point of $\mathbb{R}^2$ with coordinates x_1 and x_2 . The transformation above is then read as

$$X_1 + X_2 i = (a + bi)(x_1 + x_2 i) + (d_1 + d_2 i),$$

or

$$\xi = \alpha x + \delta,$$

40 where the letters represent complex numbers. In this way the two-dimensional scale of complex numbers becomes a group under addition and multiplication, just as the one-dimensional scale did. But the former yields the advantage of greater generality; first of all, any point is a power of any other point. Therefore the projective, as well as the Cartesian, "space is often built up from $n + 1$ (n respectively) complex scales instead of real scales, as we did above.

Further, by adding in the usual way an " $n-1$ space at infinity" one can complete Cartesian "space to the connection of a projective "space. Of course this way of completing is completely arbitrary; one might as well add a sphere at infinity, as it is sometimes useful in potential theory, or again, one might add a single point at infinity, thereby making the Cartesian space into a two-sided closed space.

Projective definition of the group of complex operations

If we do the latter, in particular for a plane, then we can map it on an oval quadric in ordinary space in such a way that the group of complex operations appears as *the group of those projective transformations of the space, which leave invariant the oriented quadric with a single point* (namely the point corresponding to the 'point at infinity' of the Euclidean plane). In this way the group of complex operations can be defined independently of the Euclidean congruence group.¹⁾

Group of complex projective transformations

The group of complex operations, considered on an oval quadric, permits an extension to the *general projective transformation group of that quadric with invariant orientation*; mapping back on the Euclidean plane we obtain the *conformal transformation group of the plane with invariant orientation*, which again can be read as the *projective transformation group of the complex scale*.

Characterization of the projective group

The group of projective transformations in n -space can be characterized after LIE as the *finite continuous* (i.e. generated by continuous change of a finite number of parameters) *Lie group, for which $n+3$ (and no smaller number) of different points have an invariant*.²⁾

The non-Euclidean groups

Selecting from the projective group those transformations which leave a quadratic $n-1$ -space invariant³⁾ we find the so-called *non-Euclidean groups of n -space*. For two points any function of their anharmonic ratio with respect to the quadric remains invariant. As 'distance' one chooses that function which is additive along the straight lines, i.e. the logarithm, multiplied by some constant. For this choice the straight lines turn out to be geodesics. Since it is not desired to admit lines on which the distances would reduce to 0, there remain only 3 kinds of fundamental $n-1$ -quadrics, namely the general oval, which in its interior gives rise to

¹⁾ By this method of defining a set of operations as a group (which can also be applied to quaternions and to higher complex numbers) the mapping of the parameter continuum on the transformed continuum plays a secondary part. In the case of the ordinary complex operations the group definition avoids the necessity of letting some point and some line play a special part as the zero point and the line of reals.

²⁾ Here and elsewhere LIE supposes that his groups are generated by differentiable infinitesimal transformations. Sometimes he even supposes them to be analytic. In this respect his theories are open to many improvements. See also p. 51 [p. 39].

³⁾ If one wishes to keep an $n-1$ -space invariant and still retain a projective group with $\frac{1}{2}n(n+1)$ (the number for the Euclidean congruence group) parameters, one can take for this invariant space nothing else than one of the first or second degree, as LIE has shown (Theorie der Transformationsgruppen III, with reference to an investigation of KLEIN and LIE in Math. Annalen 4 (1871), p. 50–84 [see F. Klein, Ges. math. Abh., p. 424–459]).

hyperbolic geometry, the general imaginary (with real equation), which gives rise to elliptic geometry, and as a transition between these two the imaginary $n-2$ quadric, giving rise to Euclidean geometry. Of these three the hyperbolic and the Euclidean group can be mapped on the (not completed) Cartesian space; and as uniform continuous groups the Euclidean and the hyperbolic group have a discontinuity which divides them into two subgroups, called *groups of motions*.

[[11]]

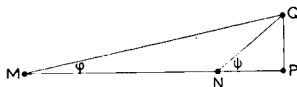
Deduction of the differential of the arc for the Euclidean and the non-Euclidean groups

It is possible in different ways to characterize together the Euclidean and non-Euclidean motions. In the first place one can start by considering in general an "space where the distance between two adjacent points is defined by a positive definite expression of the form $\sqrt{\sum f_{pq}(x_1, \dots, x_n) dx_p dx_q}$ with determinant different from 0 (the f twice differentiable functions), in other words, where Euclidean geometry is valid locally (which, for instance, is always the case for curved spaces, imbedded in higher Euclidean spaces). We then wish to find a transformation group leaving invariant the elements of distance, consequently also the geodesics. We require that every point can be transformed into every other point; that after fixation of a single point P_1 a second point P_2 can still move freely on a 'geodesic $n-1$ sphere' around P_1 , in such a way that the geodesic $n-1$ spheres around P_1 completely fill up the space; that, after the fixing of P_2 , a third point P_3 on that geodesic sphere moves freely on an $n-2$ sphere around P_2 , while these $n-2$ spheres around P_2 completely cover the above-mentioned $n-1$ sphere, etc.

We solve this problem first for a ^{2} space, choosing as coordinates the geodesic lines starting from a point M and the geodesic circles around M . Because of the free mobility around M the arc length ds must be of the form $\sqrt{dr^2 + f(r)^2 d\varphi^2}$. From this we find the general differential equation of the geodesic lines:

$$ds = \frac{f(r)^2}{c} d\varphi.$$

44



In the figure MP , MQ and NQ are geodesic lines, PQ is a small geodesic arc around M , and the angles φ and ψ are very small, thus for NQ c is very small. We put $MN = r_1$; $MP = r_2$, and we have:

$$\psi = \frac{f(r_1) d\varphi}{ds} = f(r_1) \frac{c}{f(r_1)^2} = \frac{c}{f(r_1)}.$$

[[34]]

Further:

$$\varphi = c \int_N^Q \frac{ds}{f(r)^2} = c \int_{r_1}^{r_2} \frac{dr}{f(r)^2}$$

and from this:

$$PQ = cf(r_2) \int_{r_1}^{r_2} \frac{dr}{f(r)^2}.$$

As the plane must be structured around N in the same way as around M, we must also have

$$PQ = \psi \cdot f(r_2 - r_1).$$

$$f(r_2 - r_1) = f(r_1) \cdot f(r_2) \cdot \int_{r_1}^{r_2} \frac{dr}{f(r)^2}. \quad (H)$$

From this, taking ε to be very small,

$$\begin{aligned} \int_{\varepsilon}^{r_1} \frac{dr}{f(r)^2} &= \frac{f(r_1 - \varepsilon)}{f(r_1)f(\varepsilon)} = \\ &= \frac{1}{f(\varepsilon)} - \frac{\varepsilon}{f(\varepsilon)} \cdot \frac{f'(r_1)}{f(r_1)} + \frac{\varepsilon^2}{2f(\varepsilon)} \cdot \frac{f''(r_1)}{f(r_1)} \dots \end{aligned}$$

Similarly

$$\int_{\varepsilon}^{r_2} \frac{dr}{f(r)^2} = \frac{1}{f(\varepsilon)} - \frac{\varepsilon}{f(\varepsilon)} \cdot \frac{f'(r_2)}{f(r_2)} + \frac{\varepsilon^2}{2f(\varepsilon)} \cdot \frac{f''(r_2)}{f(r_2)} \dots \quad 45$$

By subtraction from the last two equations:

$$\begin{aligned} \int_{r_1}^{r_2} \frac{dr}{f(r)^2} &= \frac{\varepsilon}{f(\varepsilon)} \cdot \left\{ \frac{f'(r_1)}{f(r_1)} - \frac{f'(r_2)}{f(r_2)} \right\} \dots \\ &= \frac{f'(r_1)}{f(r_1)} - \frac{f'(r_2)}{f(r_2)} \dots \end{aligned}$$

But also

$$\int_{r_1}^{r_2} \frac{dr}{f(r)^2} = \frac{f(r_2 - r_1)}{f(r_1)f(r_2)}.$$

Thus

$$f(r_2 - r_1) = f'(r_1)f(r_2) - f'(r_2)f(r_1). \quad (I)$$

Now $f(\varepsilon) = \varepsilon + \dots$, so we know that

$$\begin{aligned} f(0) &= 0. \\ f'(0) &= 1. \end{aligned}$$

Substituting $r_1 = \varepsilon$ in (1) we obtain:

$$f(r_2) - \varepsilon f'(r_2) = f(r_2) + \varepsilon f''(0)f(r_2) - \varepsilon f'(r_2). \quad (2)$$

So $f''(0) = 0$.

Let us differentiate (1) with respect to r_1 , obtaining

$$f'(r_2 - r_1) = f'(r_1)f'(r_2) - f''(r_1)f(r_2). \quad (3)$$

If we put $r_1 = \varepsilon$ in this equation, we obtain

$$\begin{aligned} f'(r) - f''(r_2)\varepsilon &= f'(0)f'(r_2) + \varepsilon f''(0)f'(r_2) \text{ (cancelled)} - \\ &\quad - f''(0)f(r_2) \text{ (cancelled)} - \varepsilon f'''(0)f(r_2). \\ f''(r_2) &= f'''(0)f(r_2). \end{aligned} \quad (4)$$

46

If we put $f'''(0) = a$, then we have for f the differential equation:

$$\frac{d^2 f}{dr^2} = af,$$

which has three groups of solutions, depending upon the sign of a , namely

$$\begin{aligned} \text{(I)} \quad f &= c_1 \sin(\alpha r + c_2), \\ \text{or, because } f(0) &= 0 \text{ and } f'(0) = 1, \end{aligned}$$

$$f = \frac{1}{\alpha} \sin \alpha r.$$

$$\begin{aligned} \text{(II)} \quad f &= c_1(r + c_2), \\ \text{or, because } f(0) &= 0 \text{ and } f'(0) = 1, \\ f &= r. \end{aligned}$$

$$\begin{aligned} \text{(III)} \quad f &= c_1 \operatorname{sh}(\alpha r + c_2), \\ \text{or, because } f(0) &= 0 \text{ and } f'(0) = 1, \end{aligned}$$

$$f = \frac{1}{\alpha} \operatorname{sh} \alpha r.$$

All three solutions turn out to satisfy (H). If we desire only surfaces without singularities, then (II) gives the ordinary Euclidean plane, (III) the hyperbolic plane and (I) either a sphere (two-sided) or an elliptic plane (one-sided, having the connection of the projective plane).

Finally these four surfaces turn out to have the required homogeneous structure.

Let us now investigate the ³spaces which satisfy the conditions of the problem. We know that the geodesic spheres around a point must admit free mobility in themselves; but as these spheres are firstly finite and secondly two-sided we can

take for them, out of the four surfaces found above, only ordinary spheres. Investigating which function of the radius the arc lengths on these spheres can be, we find first that the geodesic lines lie in *planes through M* (here defined as surfaces generated by the geodesics from M to a great circle on a sphere around M), and then, that again only the three functions mentioned above, and consequently only the four ³spaces constructed from them, are possible.

In the same way as from 2 to 3 the transition from 3 to 4 dimensions is effected, etc. For any number of dimensions there exist only the four types of spaces mentioned above, in other words, there are no other types of transformation groups satisfying the conditions of the problem.

Of the two groups corresponding to the arc length I the spherical can be applied to a Cartesian space completed with a single point, and the elliptical to a Cartesian space completed with an $n-1$ space.

These characterizations of the groups of Euclidean and non-Euclidean motions are very near to those originally given by RIEMANN ¹⁾. The assumptions that the arc length is quadratic, and that the coefficients are differentiable, are somewhat arbitrary. LIE ²⁾ has weakened the conditions a little by characterizing the Euclidean and non-Euclidean motions as those Lie-groups which firstly leave invariant a quadratic arc length with non-zero determinant, and secondly transform the $n-1$ directions through a fixed point in such generality as is admitted by the invariance of the equation obtained by putting the arc length equal to 0 (which of course determines a quadratic space in the sheaf of line-elements). If the first condition is replaced by the requirement that the equation which puts the arc length equal to 0 must remain invariant, we obtain in addition: firstly the group consisting of Euclidean motions and similarity transformations, and secondly that consisting of Euclidean motions, similarity transformations and transformations on reciprocal radii (the *conformal group*).

Characterization of the Euclidean and non-Euclidean groups by HELMHOLTZ

HELMHOLTZ ³⁾ has tried to free the characterization from the existence of an arc length. He investigated in an n space all the invertible continuous groups with $\frac{1}{2}n(n+1)$ parameters, such that the functions which represent the transformations admit a certain number of derivatives (namely so many as he needs for his considerations), that two points have one single invariant, giving for every pair of points a *real* restriction, that more than two points have no new invariant, and

¹⁾ Ueber die Hypothesen, welche der Geometrie zu Grunde liegen. Abh. Königl. Ges. Wiss. Göttingen 13 (1868), p. 1; Gesammelte Werke 1e Aufl. p. 254, 2e Auflage p. 272–287.

²⁾ Theorie der Transformationsgruppen III, Abt. IV, Kap. 17.

³⁾ Ueber die Tatsachen, die der Geometrie zu Grunde liegen, Göttinger Nachr. 1868, p. 193–221; Ges. Wiss. Abhandl. II, p. 618–693.

finally that these invariants express the only restrictions to the freedom of motion of the points.¹⁾

Finally he adds the so-called *monodromy postulate*, requiring that after fixation of $n-1$ points an n th point can still describe a *periodic* orbit.

Correction of HELMHOLTZ' characterization by LIE

He then proves that only the Euclidean and the non-Euclidean motions satisfy all these conditions. However, LIE has shown later on²⁾ that:

1°. HELMHOLTZ' argument is invalid, i.e. he tacitly introduces more conditions. These come to the same thing as saying that from the assumption of his conditions for pairs of points at a finite distance from each other he derives their validity for pairs of points infinitely near to each other; LIE shows by counterexamples that this conclusion is faulty in respect of the invariant of two points, free mobility and monodromy.

He shows further how HELMHOLTZ' calculations become correct when his conditions for pairs of points at a finite distance from each other are replaced by others, valid for pairs of points which are infinitely near to each other.

2°. That after this more correct formulation of HELMHOLTZ' axioms they still contain redundant components. Namely, LIE shows that the following characterization is sufficient: An invertible Lie-group admitting in one single point of general position *free mobility in the infinitesimal* (i.e. the possibility of continuous motion after fixing the point, a line-element through the point, a plane element through the line-element, a ^{3} element through the plane element, ... and an $^{n-2}$ -element through the $^{n-3}$ element, but no longer if in addition an $^{n-1}$ element through the $^{n-2}$ element must remain fixed). Only when $n = 2$ does this characterization give in addition the group of spiral transformations, which we ought to exclude by a special axiom, for instance by HELMHOLTZ' monodromy postulate.

3°. That even if one wishes to characterize the groups, as HELMHOLTZ originally tried, by the behaviour of pairs of points at finite distance, his formulation contains too much. LIE proves that the following characterization is sufficient: 'An invertible Lie-group such that after fixing one point another point can still move freely in a so-called pseudo-spherical space, which does not contain the fixed point and which has in general $n-1$ dimensions. Further within some finite domain, after fixation of $q < n-1$ points, a point of general position can still move freely in the intersection of the pseudo-spherical spaces determined by the fixed points.'

¹⁾ LIE (Theorie der Transformationsgruppen III, pp. 487 and 505) has made the remark that the existence of a 'distance' (i.e. an invariant) for two points at a finite distance from each other, and the existence of an invariant element of arcs (i.e. of an invariant 'length' for every curve) are two mutually independent conditions.

²⁾ Leipziger Berichte 42 (1890), p. 284-321, 355-418; Theorie der Transformationsgruppen III, Abt. V.

The characterization of the plane congruence group independent of differentiability
by HILBERT

It is natural to try, for each of the above-mentioned characterizations of groups, to free them from the restriction to Lie-groups ¹⁾, more in general to aim at the construction of a theory of groups independent of postulates on differentiability (cf. HILBERT, *Mathematische Probleme*, Vortrag gehalten auf dem internationalen Mathematiker-Congresse Paris 1900, Problem no. 5, *Göttinger Nachrichten* 1900, p. 269 [Ges. Abhandlungen III, p. 290–329]), as we have carried it through above for a most simple case, namely the group of fundamental operations with natural numbers.

[[14]]

52

Another case that has been settled concerns the group of Euclidean or non-Euclidean motions in the plane. HILBERT ²⁾ has characterized them as *the group of invertible uniform transformations leaving invariant the connections of incidence and forming a closed system* (i.e. if there are transformations in the group which make a definite set of points approximate another definite set of points, then there is also a transformation in the group transforming the first set into the second), *and for which every 'circle'* (i.e. the set of points into which, after fixing one point, another point can still be transformed) *consists of an infinity of points.* ³⁾

From this he proves first that every circle is a closed continuous curve without double points (a closed 'Jordan curve'), that the circles around a point, enclosing each other, fill up the whole plane, and that by a rotation of the plane around P all the points together describe their circles according to functions of one continuous parameter; further that any point can be transformed by a motion into any other point and that after fixing two points the whole plane is fixed. The second part of his argument then serves to introduce straight lines. Between two points he constructs the straight line connecting them by constructing the dense scale of middle points, where he means by the middle point of AB the centre of the rotation that transforms A into B and B into A (so-called half-turn), and he proves that this sequence of points in the plane has no gaps, and consequently together with its limit points forms a continuous curve. Further the straight line

53

¹⁾ KLEIN (*Zur ersten Verteilung des Lobatchefsky-Preises*, *Math. Annalen* 50 (1898), p. 583–600; [Ges. math. Abh. I, p. 384–401]) defends the restriction to differentiable functions in LIE's investigations on the foundations of geometry by the remark that any empirical function can be represented by an analytical function with as good an approximation as we desire. But a priori it is very well conceivable that there exist non-Lie groups such that the approximating analytical transformations do not constitute a group (though they 'almost' constitute a group, i.e. they approximate the group property without being able to attain it as long as they remain analytical).

²⁾ *Math. Annalen* 56 (1903), p. 381–422 [Grundlagen der Geometrie, Anhang IV]. In this case as well the analytical character of the functions turns out to be a *consequence* of the group property.

³⁾ However, because he does not complete the Cartesian plane either by a point or by a line at infinity, he finds only the Euclidean and the hyperbolic groups of motions.

is produced beyond its endpoints by performing half-turns around these endpoints; thereupon it is proved that two straight lines have at most one point in common and that two points are always connected by a straight line, and of course not by more than one. Finally the theorems on congruence are proved, whence, depending on the parallel postulate added, Euclidean or hyperbolic geometry can be built up.

Characterization of the linear system of the elliptic or projective space

One can ask, given a space having the connection of a projective space, for the characterization of a system of curves, surfaces, spaces etc., having the properties of the projective system of straight lines, planes, plane ³spaces etc. The following is sufficient:

- 54 Two points always determine one straight line; through a straight line and a point outside it there goes a plane which contains every straight line having two points in common with it; through a plane and a point outside it there goes a ³space containing every straight line that has two points in common with it; etc.; further: a straight line and an $n-1$ space have a point of intersection.

It is easy to deduce from these properties the unicity of the quadrangle construction for the 4th harmonic in a line of points, in a pencil of lines, etc. Now choose arbitrarily $n+1$ points, and let them correspond with the $n+1$ basis points of a projective n space, constructed as above; then assign an arbitrarily chosen $(n+2)$ th point to the unit point of the projective space. By successive quadrangle constructions (that is, solely by projections and intersections) one can construct, from the arbitrarily chosen $n+2$ points, enough points to ensure that every point in the projective n space with rational coordinates corresponds with one of them. It follows from a proof of LÜROTH and ZEUTHEN, communicated by KLEIN (Math. Annalen 7 (1874), p. 531–537 [Gcs. math. Abhandlungen I, p. 344–350]), that the points constructed in this manner lie everywhere dense. And from this it follows that the one-one correspondence between the points of the two spaces holds for the irrational points as well. ¹⁾

- 55 *Characterization of the linear system of the Cartesian or Euclidean and of the hyperbolic space*

A system corresponding to the straight lines, planes, ³spaces etc. of a Euclidean or hyperbolic n space (in both cases the class of points fills a Cartesian n space) can be characterized as follows:

Two points always determine one straight line; through a straight line and a point outside it there passes a plane, containing every straight line that has two

¹⁾ It seems that by this mapping on a projective space that had been constructed beforehand, KLEIN's developments (Vorlesungen über nicht-Euklidische Geometrie, Göttingen 1893, p. 319–353 [2. Auflage, bearbeitet von W. Rosemann, Berlin 1928, p. 153–163]), aiming at a direct proof of further theorems from the axioms of projective geometry, become superfluous.

points in common with it; through a plane and a point outside it there passes a ${}^3\text{space}$ containing every straight line that has two points in common with it, etc.; further: every $n-1$ space divides the n space in two parts in such a way that every straight line connecting a point of one part with a point of the other part, intersects the $n-1$ space in a point lying between these two points. In fact, following SCHUR (Math. Annalen 39 (1891), p. 113–124), one can prove that this system can be completed by its ‘ideal elements’ to the projective system considered above. If, in addition, we require that EUCLID’s axiom of parallels holds, then necessarily the linear system of Cartesian n space (or of projective n space stripped of a single plane $n-1$ space) turns up; if however we require LOBATCHEFSKY’s axiom of parallels, then one obtains the linear system of a Cartesian (or also of a projective) space, lying inside an arbitrary convex oval, which can be joined to the given system as its ‘oval of limitpoints’.

56

Thus we have the linear system of hyperbolic n space only if the oval of limitpoints is a quadratic $n-1$ space; this can be enforced by requiring that the given system admits a projective uniform group with $\frac{1}{2}n(n-1)$ parameters.

Variational problems leading to linear systems

Now one may ask how to determine a curve between any two points of a Cartesian space in such a way that these curves satisfy the characterizations of a linear system as given above; one will think firstly of variational problems for which these curves are extremals.

We know that the required system of curves can be mapped uniformly either on the straight lines of a Cartesian space completed by a space at infinity, or on those simply of a Cartesian space, or on those of a Cartesian space lying inside a convex oval.

Consequently one looks for variational problems in Cartesian space which give as their extremals the straight lines, for in the most general uniform continuous transformation of the solution of this problem one finds the general variational problem for which the extremals possess the same incidence properties as the linear system.

57

Investigations on this problem have been made by HAMEL (Math. Annalen 57 (1903), p. 231–264) for the plane, and in outline for ${}^3\text{space}$; we shall précis them for the plane. HAMEL puts the question from a somewhat different point of view than is taken here; by so doing he subjects the required integral to some restrictions, so that it more or less coincides with the ordinary conception of length in metrical geometry, and so that the extremals which he looks for are more specially *minimal* curves.

Thus he looks for minimal curves of the integral

$$\int ds \cdot f(x, y, \text{tg } \vartheta) \equiv \int dx \cdot g(x, y, \text{tg } \vartheta),$$

where f denotes a positive, single-valued (because the element of length must be invertible), and in general continuous function, differentiable with respect to each of its three arguments. As the differential equation of the variational problem must take the form

$$\frac{d^2 y}{dx^2} = 0,$$

g must satisfy the differential equation:

58
$$\frac{\partial^2 g}{\partial p \partial x} + p \frac{\partial^2 g}{\partial p \partial y} - \frac{\partial g}{\partial y} = 0 \quad \left(\text{putting } \frac{dy}{dx} = p \right).$$

By differentiation of this equation with respect to p one obtains an equation in $\frac{\partial^2 g}{\partial p^2}$, with the general solution:

$$\frac{\partial^2 g}{\partial p^2} = W(p, y - px);$$

from which we finally find for g :

$$g = \int_c^p \int_c^p W(p, y - px) dp dp + \frac{\partial u}{\partial x} + p \frac{\partial u}{\partial y}$$

(u being an arbitrary function of x and y).

Remarking that

$$\int_c^p \int_c^p v(p) dp dp = \int_c^p (p - \xi) v(\xi) d\xi$$

and putting $W = \cos^3 \vartheta \cdot w$, we obtain:

$$\begin{aligned} g &= \int_{\vartheta_0}^{\vartheta} \cos \tau \cdot w(\operatorname{tg} \tau, y - x \operatorname{tg} \tau) (\operatorname{tg} \vartheta - \operatorname{tg} \tau) d\tau + \frac{\partial u}{\partial x} + \operatorname{tg} \vartheta \frac{\partial u}{\partial y}. \\ g &= \int_{\vartheta_0}^{\vartheta} \frac{1}{\cos \vartheta} \cdot \sin (\vartheta - \tau) \cdot w \cdot d\tau + \frac{\partial u}{\partial x} + \operatorname{tg} \vartheta \frac{\partial u}{\partial y}. \\ f &= \int_{\vartheta_0}^{\vartheta} \sin (\vartheta - \tau) \cdot w \cdot d\tau + \frac{\partial u}{\partial x} \cos \vartheta + \frac{\partial u}{\partial y} \sin \vartheta. \end{aligned}$$

Here the condition that we have a minimum is that

w is positive for every x, y and ϑ ;

whilst, because f is single-valued, there follows further:

a) w is single-valued.

$$b) \frac{\partial u}{\partial x} = \frac{1}{2} \int_{\vartheta_0}^{\vartheta_0 + \pi} \sin \tau \cdot w \cdot d\tau.$$

$$\frac{\partial u}{\partial y} = \frac{1}{2} \int_{\vartheta_0}^{\vartheta_0 + \pi} \cos \tau \cdot w \cdot d\tau.$$

Thus finally the arc length becomes:

$$dl = \frac{ds}{2} \int_{\vartheta - \pi}^{\vartheta} \sin(\vartheta - \tau) \cdot w(\tau) \cdot d\tau.$$

Now HAMEL first asks when the straight lines of the plane, including the line at infinity, belong to the minimal curves – so that the variational problem determines a *projective* linear system –; he finds as the condition that $\int dl$ remains finite along every straight line and has the same value along every straight line.

Hereupon he gives a category roughly coinciding with a geometry which MINKOWSKI constructed in his 'Geometrie der Zahlen', where every straight line, but not the line at infinity, belongs to the minimal curves – thus the variational problem determines here a complete Euclidean linear system –; he supposes w

to be independent of its second argument, and $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial z}$ to be constant, but relaxes the invertibility of the arc length. In this way he finds as calibration-curve, i.e. the set of points at distance 1 from the origin:

$$F(r, \vartheta) = 1 - r \left\{ \int_{\vartheta_0}^{\vartheta} \sin(\vartheta - \tau) \cdot w(\tau) \cdot d\tau + \alpha \cos \vartheta + \beta \sin \vartheta \right\} = 0.$$

$$\frac{d^2 \left(\frac{1}{r} \right)}{d\vartheta^2} + \frac{1}{r} = w(\vartheta).$$

And, as the first member of this equation, multiplied by a positive factor, is the radius of curvature, while the only restriction on F is that w must be everywhere positive, he finds for the only restriction on the calibration-curve that it is everywhere convex.

Finally he gives as an example of a solution which represents a Euclidean linear system inside a convex oval, the so-called Hilbert geometry: in Math. Annalen 46 (1895), p. 91–96 [Grundlagen der Geometrie, Anhang I] HILBERT defines the distance between two points as the logarithm of the anharmonic ratio

60
[[16]]

which they form with the points of intersection of their connecting line with a convex oval, inside which they lie; he shows that by this definition the sum of two sides of a triangle is greater than the third side. HAMEL shows now how this geometry as well can be derived from his general formula.

For this purpose he tries to determine functions W and u such that

$$61 \quad \int_{x_1}^{x_2} dx \int_c^p dp \int_c^p dp W + u(x_2, y_2) - u(x_1, y_1) = \log \frac{(x_1 - v_1)(x_2 - v_2)}{(x_2 - v_1)(x_1 - v_2)},$$

and he finds, putting $y - px = b$,

$$W[p, y - px] = \left[\frac{\partial^2 (v_2 - v_1)}{\partial b^2} \right]_p$$

for which expression he easily proves that it is positive (negative) for ϑ in the first or third (second or fourth) quadrant; from this it follows that in every case w is positive, as is required for the minimum property.

He finds further that

$$u = \log \frac{x - v_2(c, y - cx)}{x - v_1(c, y - cx)}.$$

We can produce the Minkowski geometry mentioned above (with an invertible arc length) as a special case of this Hilbert geometry.

MINKOWSKI takes the arc length $ds f(\vartheta)$, and HILBERT

$$ds \left\{ \frac{1}{s - s_1} + \frac{1}{s_2 - s} \right\};$$

if we let Hilbert's limiting oval tend to infinity, preserving its shape, then $\frac{1}{s - s_1} + \frac{1}{s_2 - s}$ becomes infinitely small, but at the same time equal for parallel directions, so after multiplication by an infinite constant it becomes a function of the direction alone, and the arc length assumes the form $ds f(\vartheta)$.

[[17]]

The possible pointsets

In the beginning of this chapter we have constructed two sorts of linearly ordered sequences of points, namely the order type ω of the enumerated positive ordinal numbers and the inverse type ω^* ; also we have the everywhere dense denumerable sequence (the ordertype η of the rationals, all of them, or between 0 and 1, or also the dual scale, complete or between 0 and $1^{(1)}$). Moreover, we have considered the

¹⁾ where we shall be free to include 0 and 1 or not, as we please, in the sequence of points.

intuitive continuum as a measurable continuum, and we have seen that every point on it admits an approximation by a dual scale. However, the *continuum as a whole* was given to us by intuition; a construction for it, an action which would create from the mathematical intuition ‘all’ its points as individuals, is inconceivable and impossible.

The mathematical intuition is unable to create other than denumerable sets of individuals. But it is able, after having created a scale of ordertype η , to superimpose upon it a *continuum as a whole*, which afterwards can be taken conversely as a measurable continuum, which is the matrix of the points on the scale.

It follows that a given continuum can be covered with gaps by another continuum; for this purpose it is sufficient to construct on the first continuum an order type η which does not cover it everywhere densely, and then to construct the continuum over that order type η ; we can always identify a point on the second continuum with the limit point on the first continuum of its approximating sequence. In this sense we can say: ‘The points of the second continuum are a part of those of the first one’; and in this sense we have now three methods for the construction of ‘pointsets on the continuum’, namely:

1°. We can construct on the continuum discrete, individualized sets of points which are finite, of order type ω , of order type η , or can be obtained from such sets of points by alternation or subordination. 1) The number of these points is always denumerable, and likewise the number of the intervals determined on the continuum by pairs of points from the set is denumerable. In each of these intervals, and also in its totality, the set may be dense or not (by *dense* we mean: of the order type η after every well-ordered or inversely well-ordered subset has been contracted to a single point).

We can also say: in any segment of the continuum (and also in the whole continuum) the set may be dense or not; a closer investigation shows that the latter

1) If we use only the order type ω in combination with finite sets, then we obtain the so-called well-ordered sets. In such a set an element that is introduced later in the construction, comes also later in the order. Of course, in the construction one may take as elements of the finite sets, or of the sets of ordertype ω , all the well-ordered sets that have been previously constructed. In this way we obtain successively 1, 2, ..., ω , $\omega+1$, $\omega+2$, ..., $\omega\cdot 2$, ..., ω^2 , ω^2+1 , ..., ω^3 , ..., ω^ω , ..., ω^{ω^ω} , ..., ε_1 (i.e. $\omega^{\omega^{\omega^{\dots}}}$, exponentiation being iterated ω times), ...

A well-ordered set has the property that it and each of its subsets has a first element. Besides the well-ordered sets we can also construct inversely well-ordered sets. The only way to construct sets which are not well-ordered or inversely well-ordered is by ω times repeated insertion of well-ordered sets in well-ordered sets; by this process dense (see above) subsets can be constructed.

Compare in this connection a theorem of BERNSTEIN (Math. Annalen 61 (1905), p. 144):

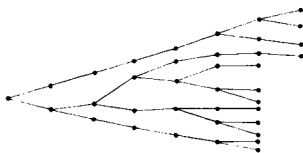
Any ordered denumerable set (and consequently any individually constructible set, for even if in the construction I say nothing about order, I can tacitly consider what is constructed later on as following after what had been previously constructed) can be mapped on a part of the order-type η preserving the order relation.

63

64
[[18]]

[[19]]

distinction works out as follows if we approximate the set by a dense dual scale which has been constructed on the segment under consideration as a unit segment: in determining each dual digit the preceding ones either determine it or they leave open the choice between two digits; in the latter case, the next digit is again either determined or can be chosen from two digits, etc. This can be illustrated by a drawing of the form:



and so on.

[[20]]

66

If in this structure we cut off every branch which ceases to branch ¹⁾ we are left either with nothing or with a continually multiplying dual branching. In the latter case the set is dense in the interval under consideration, in the former it is not.

2°. In the intervals where the set constructed under 1° is dense, we can transform it by the contractions described above into an everywhere dense set, and then apply to this set the operation of 'completion to a continuum': it is always possible to define clearly the intervals which we select for this operation, for, as their number is denumerable, they are individualized.

3°. We can construct a set of points by deleting from a continuum a dense scale constructed on it in some interval.

Solution of the continuum problem

If now in the construction of a set of points the operation 2° (whether or not in combination with 3°) has been applied, it can be 'mapped' on a continuum. This must be understood as follows: in both sets (the continuum and the given set) a well defined, and consequently denumerable, pointset is chosen in such a way that all the other points can be obtained by approximation from everywhere dense parts of this set. Then the undefined points are brought into one-to-one correspondence by mapping the everywhere dense parts, with respect to which the infinitely proceeding approximations must be taken, onto each other; then the points which were defined can still be brought into correspondence with each other, because they are denumerable in both sets.

67

It follows that every set of points on the measurable continuum (and consequently also on the simple intuitive continuum, which in fact we can only handle

¹⁾ from left to right according to the rank of the dual digit of the point of ramification; after the cutting operation is achieved for all ω ranks, it is applied again to the residue that was left.

after having made it measurable ¹⁾ – or built up out of individualized measurable parts) which is not denumerable, has the power of the continuum.

Thus the ‘continuum problem’, put forward by CANTOR in 1873 [G. Cantor, J. reine angew. Math. 77 (1874), p. 258–262, Ges. Werke p. 115–118, proved that the continuum is not denumerable. In J. reine angew. Math. 84 (1878), p. 242–258, Ges. Werke p. 119–133 he conjectured that the cardinal number of the continuum follows immediately after the denumerable.] and mentioned by HILBERT (Mathematische Probleme, Problem no. 1, p. 263 [Göttinger Nachr. 1900, p. 253–297; Ges. Abhandlungen III, p. 290–329]) as still open, seems to be solved, by keeping strictly to the view that a continuum as a set of points must be considered with respect to a scale of ordertype η .

Non-Archimedean uniform groups on the one-dimensional continuum

We can imagine a Cartesian space of ω dimensions, or of $\omega^* + \omega$ dimensions, and consider in it all the points whose coordinates of index less than a given number are zero. ²⁾

We can go further and imagine a Cartesian space of $(\omega^* + \omega)^n$ dimensions. Then every coordinate carries as indices n positive or negative integers, and we restrict ourselves to those points whose coordinates are different from zero only if, given the preceding indices, every index is greater than a given number, so that the numbers of the remaining coordinates form a well-ordered set.

These points can be linearly ordered by defining the order relation for two points by means of the coordinate of least number in which they are different. Now the problem is to construct on such a pseudo-continuum (which is an ordinary measurable continuum with all sorts of points between an arbitrary point on it and a segment having that point as its boundary, and in addition with all sorts of points to the left and to the right of all points of the ordinary continuum) a group of operations which for the ordinary continuum yields a group of arithmetical operations. We can start by constructing for each of the coordinates a scale and from that an ordinary group of additions. In that way we construct at the same time a group of additions for the whole space, which is associative and commutative, and by which any point can be transformed into any other point.

If we choose on each of the scales a 1-point, then the operation of ‘multiplication by a point of the ordinary continuum’ (i.e. by a point for which every coordinate is zero except that with number 0) is self-explanatory; it is associative and

¹⁾ The non-Archimedean pseudo-continua, which will be discussed below, must be considered, according to our conception of the continuum, as higher-dimensional continua (ordered in a special way and subject to a special group of transformations); however according to CANTOR higher-dimensional, and even denumerably-infinite-dimensional continua, can be mapped on one-dimensional continua, so the theorem holds also for higher-dimensional as well as for non-Archimedean continua.

²⁾ Compare in this connection HILBERT, Festschrift zur Feier der Enthüllung des Gauss-Weber-Denkmal in Göttingen [Grundlagen der Geometrie], § 33.

distributive with respect to the group of additions. Let us look for an associative operation which increases by 1 the numbers of the coordinates and which we shall denote by ' $a_1 \times$ ' in particular ' $l_1 \times$ ' if it always maps 1-points to each other (for which purpose we can always make an appropriate choice of our 1-points). As it must transform the groups inside the single coordinates into each other, we have generally

$$l_1 \times a_\alpha = a_{\alpha+1}.$$

Let us look further for an associative operation which increases all the numbers of coordinates by ω , and which we shall denote by ' $a_\omega \times$ ', in particular by ' $l_\omega \times$ ' if it transforms the 1-point of the 0-coordinate into that of the ω -coordinate, that of the ω -coordinate into that of the 2ω -coordinate, etc.

As, again, it must transform the groups inside the single coordinates into each other, we have, if α is a finite number:

$$l_\omega \times a_\alpha = \{p_\alpha a\}_{\omega+\alpha}.$$

As the operations l_1 and l_ω must be associative in their combinations, we have further:

$$\begin{aligned} \{p_2 a\}_{\omega+2} &= l_\omega \times a_2 = (l_\omega \times l_1) \times a_1 = p_1 \times l_1 \times l_\omega \times a_1 = \\ &= p_1 \times l_1 \times \{p_1 a\}_{\omega+1} = \{p_1^2 a\}_{\omega+2}, \end{aligned}$$

70

and obtain, continuing in this way,

$$p_\alpha = p^\alpha.$$

And in general, if τ is any finite or infinite number:

$$\begin{aligned} \{p_{\tau+1} a\}_{\tau+\omega+1} &= l_\omega \times a_{\tau+1} = (l_\omega \times l_1) \times a_\tau = p_1 \times l_1 \times (l_\omega \times a_\tau) = \\ &= p_1 \times l_1 \times \{p_\tau a\}_{\tau+\omega} = \{p_1 p_\tau a\}_{\tau+\omega+1} \end{aligned}$$

so that in general

$$p_\tau = p^\alpha,$$

where α is the finite part of the transfinite number τ .

By the same method we introduce the operation $l_{\omega^2} \times$ such that

$$l_{\omega^2} \times a_\tau = \{q_\tau a\}_{\tau+\omega^2},$$

and find from the associative property firstly:

$$\begin{aligned} q_\alpha &= q^\alpha \\ q_{\beta\omega} &= r^\beta, \end{aligned}$$

where α and β are finite numbers.

We now have, if $\tau = \zeta + \beta\omega + \alpha$, where ζ contains only terms of higher than the first degree in ω :

71

$$\begin{aligned} \{q_\tau\}_{\tau+\omega^2} &= l_{\omega^2} \times l_\tau = l_{\omega^2} \times l_\alpha \times l_{\beta\omega} \times l_\zeta = q^\alpha \times l_\alpha \times l_{\omega^2} \times l_{\beta\omega} \times l_\zeta = \\ &= q^\alpha \cdot r^\beta \times l_\alpha \times l_{\beta\omega} \times l_{\omega^2} \times l_\zeta = \{q^\alpha \cdot r^\beta\}_{\tau+\omega^2}, \end{aligned}$$

so that

$$q_{\tau} = q^{\alpha} r^{\beta}.$$

That further these conditions for p_{τ} and q_{τ} , which have been found necessary, are also sufficient, is proved by calculating:

$$1_{a_1 + b_1 \omega + c_1 \omega^2} \times (1_{a_2 + b_2 \omega + c_2 \omega^2} \times 1_{a_3 + b_3 \omega + c_3 \omega^2})$$

and

$$(1_{a_1 + b_1 \omega + c_1 \omega^2} \times 1_{a_2 + b_2 \omega + c_2 \omega^2}) \times 1_{a_3 + b_3 \omega + c_3 \omega^2}.$$

We then find for both expressions

$$p^{b_1 a_2 + b_1 a_3 + b_2 a_3} \cdot q^{c_1 b_2 + c_1 b_3 + c_2 b_3} \cdot r^{c_1 a_2 + c_1 a_3 + c_2 a_3} \times 1_{(a_1 + a_2 + a_3) + (b_1 + b_2 + b_3) \omega + (c_1 + c_2 + c_3) \omega^2}.$$

Continuing in this way we can define ' $1_{\omega^3} \times$ ', ..., ' $1_{\omega^n} \times$ ' and, if we understand by multiplication by a sum, multiplication by the terms and then summation, noticing, that for the reciprocal of a number all the coordinates in succession can be approximated, so that also division can be applied, we have a complete set of arithmetical operations on our linear pseudo-continuum, for which the associative and distributive properties hold, but, if $n > 1$, not necessarily the commutative one; for instance:

$$1_1 \times 1_{\omega} = 1_{\omega + 1},$$

but

$$1_{\omega} \times 1_1 = p_{\omega + 1},$$

whereas in the case of the ordinary measurable continuum we have seen above that no group of operations can be constructed for which the commutative property fails.

Because of the lack of measurability the pseudocontinuum constructed above may be called a *non-Archimedean continuum*; moreover it lacks another property of the ordinary continuum, namely Dedekind continuity¹⁾, which admits the following formulation: If the set of points of the continuum is divided into two parts in such a way that every point of one part is higher in rank than every point of the other part, then either the lower part has a greatest point and the higher part has no least point, or the higher part has a least point and the lower part has no greatest point. In fact measurability can be immediately derived from Dedekind continuity.

The non-Archimedean continuum does possess *Veronese continuity* [G. Veronese, Grundzüge der Geometrie von mehreren Dimensionen und mehreren Arten gradliniger Einheiten, translated from the Italian, Leipzig 1894], which may be defined as follows: If the set of points of the continuum is divided into two parts in such a way that every point of one part is higher in rank than every point of the other part, and if furthermore I can choose from both parts 2 points whose

¹⁾ cf. DEDEKIND, Stetigkeit und irrationale Zahlen [Braunschweig 1872].

difference can be made smaller than any given quantity, then either the lower part has a greatest point and the higher part has no least point, or the higher part has a least point and the lower part has no greatest point. (If the condition that the difference can be made indefinitely small is not fulfilled, then it may happen that the lower part has no greatest point and the higher part has no least point.)

Non-Archimedean and non-Pascalian geometries

By the same method as we constructed projective geometry from $n+1$ ordinary continua or complex continua, provided with a group of operations, we may now do the same with non-Archimedean continua. The proofs for the linear equations of straight lines, planes etc. (where however we must remember to put the coefficients to the right of the coordinates) remain valid without change; consequently all the theorems which we have mentioned above (see p. 54 [p. 40]) as characterization of the system of straight lines, planes etc. in an ordinary projective space remain valid as well.

It follows also that DESARGUES' theorem ('If for two triangles the lines connecting corresponding vertices pass through one point, then the points of intersection of corresponding sides lie on one line') and, equivalently, the theorem on the uniqueness of the 4th harmonic point, remain valid; likewise the projectivity of harmonic pairs still holds.

74 But the theorem on the projectivity of relative coordinates turns out to depend on the validity of the commutative property of multiplication. The same is true for the equivalent PAPPUS' theorem ('If on each of two intersecting lines 3 points are given, the 3 points of intersection of the lines which crosswise connect pairs of corresponding points lie on a straight line'), which HILBERT (Grundlagen der Geometrie, in particular Chapter VI) calls PASCAL's theorem. (It can be considered as the special case for a degenerate conic of the theorem known under this name.) Thus when multiplication is not commutative, PASCAL's theorem together with several incidence theorems fail, for instance the theorem on the unicity of the common harmonic pair for two pairs of points on a straight line, and the so-called *fundamental theorem of projective geometry*: If two straight lines are related by a sequence of projectivities, then three pairs of related points determine the point corresponding with any other point.

Pascal's theorem follows from the existence of a projective congruence group.

If a *congruence group*, i.e. a group having similar properties as the elliptic, resp. hyperbolic (Euclidean) congruence group ¹⁾ exists for the non-Archimedean

¹⁾ Such a group can be characterized as a projective group which

1° can transform any point into any other point,

2° admits free mobility in the infinitesimal around every point, and determines 'spheres' around the point intersecting every halfline from the point once,

3° inverts any segment and interchanges the equal sides of any isosceles triangle;

it consists of congruent and symmetric transformations (the latter only to be mentioned expressly for the hyperbolic and Euclidean groups.)

projective geometry, or for its part inside a convex oval ¹⁾, (or in the limit inside a double counted plane n^{-1} space), then, as HILBERT has shown (Grundlagen der Geometrie, Chapter III; Neue Begründung der Bolyai-Lobatcheffskyschen Geometrie, Math. Annalen 57 (1903), p. 137–151 [= Grundlagen der Geometrie, Anhang III]; compare also TH. VAHLEN, Abstrakte Geometrie, p. 251), every incidence theorem holds, so that multiplication must be commutative.

For a non-Archimedean projective geometry inside a convex oval VAHLEN (Abstrakte Geometrie, Leipzig 1905 [[2. Auflage 1940, p. 148–169]]) has shown that every theorem on congruence, in particular Pascal's theorem, and hence commutativity of multiplication, can be derived solely from the existence of an *affine group*, i.e. a projective group which transforms any point into any other point and admits free mobility in the infinitesimal around every point.

Semi-congruent groups in non-Archimedean geometry

76

For non-Archimedean plane geometry HILBERT established still another property (Ueber den Satz von der Gleichheit der Basiswinkel im gleichschenkligen Dreieck, Proc. London Math. Society 35 (1903), p. 50–68 [Grundlagen der Geometrie, Anhang II]). If the ordinary (Euclidean or non-Euclidean) congruence groups exist, then, as we have seen, Pascal's theorem holds; but then beside these groups there exist *semi-congruent groups of motions*, possessing all the properties of the congruent groups of motions, though the plane is not symmetric with respect to that group and an isosceles triangle need not have equal angles at the base. The non-monodrome spiral group of HELMHOLTZ in ordinary Archimedean geometry has similar properties, but there a segment is not invertible, for a rotation through π changes its 'length', i.e. its invariant with respect to the group of translations; moreover, having fixed a point, any trajectory intersects every half-line emanating from that point more than once. In Hilbert's semicongruent group this is prevented by letting only the second of the two parts of the rotation angle (namely the finite and the infinitely small part; for angles there exists no infinitely large part) induce a proportional enlargement. Here every point does describe a spiral, but only with respect to the infinitely small part of the angle, which at the periods π and 2π is always 0. Thus the finite part of the distance to the centre remains constant during the rotation, so that the origin cannot be indefinitely approached; further any half-line from the centre is intersected only once by the trajectories, and finally the invertibility of segments is preserved.

77

Mathematics can deal with no other matter than that which it has itself constructed

[[25]]

In the preceding pages it has been shown for the fundamental parts of mathematics how they can be *built up* from units of perception, by simple juxtaposition,

¹⁾ If only the bounded part is given with the characterizing properties (see p. 55 [[p. 40]]), then the completion by ideal elements after F. SCHUR (Math. Annalen 39 (1891), p. 113–124) can be effected unchanged for the non-Archimedean geometry.

by building sequences of type ω or η , or by building continua, while at every stage in the process complete systems which have been constructed before can be taken as new units.

In the third chapter it will be explained why no mathematics can exist which has not been intuitively built up in this way, why consequently the only possible foundation of mathematics must be sought in this construction under the obligation carefully to watch which constructions intuition allows and which not, and why any other attempt at such a foundation is condemned to failure.

Often it is quite simple to construct inside such a structure, independently of how it originated, new structures, as the elements of which we take elements of the original structure or systems of these, arranged in a new way, but bearing in mind their original arrangement. The so-called 'properties' of a system express the possibility of constructing such new systems having a certain connection with the given system.

And it is exactly this *imbedding* of new systems in a given system that plays an important part in building up mathematics, often in the form of an inquiry into the possibility or impossibility of an imbedding satisfying certain conditions, and in the case of possibility into the ways in which it is possible.

Examples have been treated above as inquiries into the possibility of imbedding transformation groups satisfying certain conditions *into given systems*. (There we may consider the continuum of group parameters as the system to be imbedded, and the character of that continuum as a continuum of group parameters, as the condition of the imbedding.) And in this form we have given, among other things, substrata of several recent investigations which were intended to shed light upon the foundations of mathematics, and which have mathematical meaning only in the sense of our interpretation – this last on the basis of ideas to be defended in the third chapter, for which the preceding development can be considered as an illustration.

II. MATHEMATICS AND EXPERIENCE

The intellect and the jump from the end to the means

Proper to man is a faculty which accompanies all his interactions with nature, namely the faculty of taking a *mathematical view* of his life, of observing in the world repetitions of sequences of events, i.e. of causal systems in time. The basic phenomenon therein is the simple intuition of time, in which repetition is possible in the form: 'thing in time and again thing', as a consequence of which moments of life break up into sequences of things which differ qualitatively. These sequences thereupon concentrate in the intellect into mathematical sequences, not *sensed* but *observed*. And human behaviour includes attempts to observe as many of these mathematical sequences as possible, in order, whenever in the real world intervention at an earlier member of such a sequence seems more successful than at a later member, to choose the earlier one as a guide for his actions, even when his instinct is only affected by the later one. (Substitution of the *means* for the *end*.) Nevertheless the non-instinctive nature of this intellectual action renders the certainty that the parts of a sequence really belong together, anything but perfect. Consequently it can always be falsified, which is observed in the discovery that 'the rule no longer applies'.

However, in general the consideration of sequences and consequent going back from the end to the means, where intervention appears easier, show themselves very efficient tactics from which mankind derives its power. Man succeeds in discovering regularities in a limited field of phenomena independent of other phenomena which consequently can remain completely latent in the intellectual consideration of the former.

In order to maintain as long as possible the certainty of an observed regularity, one tries to *isolate* systems, i.e. to exclude those observations which disturb the regularity; in such a way man *makes* far more regularity in nature than originally occurred spontaneously; he *desires* this regularity, because it strengthens him in the struggle for life, rendering him capable of predicting, and taking action.

Mathematical systems, containing more than given reality

This intellectual manner of looking at the world expands, because man builds up pure mathematics out of the basic intuition of the intellect, without reference to immediate applicability, and thus obtains a ready supply of unreal causal sequences which only wait for an opportunity to be projected into reality. Here it must be remembered that in those mathematical systems in which no time-coordinate appears, it is found that in practical applications all their relations still become relations in time. Thus, for instance, Euclidean geometry when applied to reality

81

82

83

gives the causal relations between the results of different measurements executed with the aid of the group of the rigid bodies. – Needless to say in the application usually only a small portion of the elements and subconstructions of a mathematical system find their counterpart in reality: the others only facilitate the understanding of the whole. ¹⁾

Extension of applicability of mathematics by actual intervention

84 Similarly, after only a little development of the method, the observed sequences will no longer consist solely of phenomena observed independently of man's will, but will be complemented by phenomena evoked by man (deeds without any direct instinctive aim, but executed only to complete the causal system so that it will become more easily manageable); the simplest example of this being the sound-image ²⁾ of a cardinal number obtained by the process of counting, or the sound-image ²⁾ of the measuring number obtained by the process of measurement.

Extension by induction from the real to the possible

Mathematical science does not however derive its great power solely from the observation of sequences which are approximately equivalent for the instinct, but from combining a very large number of such sequences from one point of view by means of a mathematical system built up with the aid of mathematical induction. Such a system is called a *law*; the difference between two sequences falling within it, depends only on a difference in values of the parameters occurring in the law. ³⁾ Next to the really observed sequences with their determined parameter values, sequences with other parameter values are postulated as *possibilities*; the fact that only the observed values of the parameters occurred in reality is considered *accidental*.

On the other hand it appears that one can often successfully consider a quantity occurring in one single observed sequence as an accidental parameter value, and can thus correctly predict new sequences by means of induction.

85
[1] The simplest inductive extension to a group of possible phenomena takes place by using the coordinates of space and time as parameters. The physicist considers it accidental that a sequence realized itself *there* and *then* and not in another space and not at another time.

Continuity of functions in physics

After observing sequences and combining them by induction, the mathematical

¹⁾ In particular one often applies to observed finite sequences the extension to sequences of ω terms, which is hypothetical since it exists only mathematically; on the introduction of such a sequence depends, for example, the infinite length of the time-coordinate.

²⁾ or written symbol.

³⁾ The most important application of combining by induction is found in the causal relations between numbers, i.e. between results of counting or measuring.

action on the world proceeds further. To be able to apply a large number of processes which are dependent on the measurable continuum, the discrete observations are in the first place completed to continuous functions ¹⁾; it is not only the time-coordinate which is rendered continuous (for which ample justification is given by the intuition), but also every functional relation between measured quantities which have nothing to do with the time-coordinate. That again is an arbitrary act which is only justified because it apparently ‘works’. Rendering the observed functions continuous is done by means of the well-known method of interpolation. This is again an arbitrary act which is not refuted by practice. By interpolation one obtains analytic functions, which one is inclined to use exclusively in the study of nature anyhow. Why?

Differentiability of physical functions

Mainly because of an arbitrary act of interpreting nature anthropomorphically: one notices from the observed resistance after one’s own intervention that one can change all conditions gradually. Consequently one postulates nearly equal behaviour in closely adjacent argument values for functions in nature which can be measured in practice. ²⁾ If a function belongs to this category its difference quotients of various orders will also belong to it (for they are determined from the same measurements as the function itself); hence also the derivatives of various orders, and their difference quotients, *if these derivatives exist*. We shall show the latter to be the case. Let x_a be one of the independent variables and $\varepsilon_A(x_a)$ the maximum of the oscillations between x_a and $x_a + \Delta$ of the difference quotients corresponding to the different increments of the independent variable x_a , where the increments successively take all the values between 0 and an arbitrary small but fixed value a . Then it follows from the above postulate that ε_A approaches 0 with Δ . Furthermore we have, if p/n is a proper fraction,

$$\begin{aligned} \left(\frac{\Delta f}{\Delta x_a}\right)_{x_a} &= \frac{1}{n} \left(\frac{\Delta f}{\frac{1}{n}\Delta x_a}\right)_{x_a} + \frac{1}{n} \left(\frac{\Delta f}{\frac{1}{n}\Delta x_a}\right)_{x_a + \frac{1}{n}\Delta} + \dots + \frac{1}{n} \left(\frac{\Delta f}{\frac{1}{n}\Delta x_a}\right)_{x_a + \frac{n-1}{n}\Delta}; \\ \left(\frac{\Delta f}{\frac{p}{n}\Delta x_a}\right)_{x_a} &= \frac{1}{p} \left(\frac{\Delta f}{\frac{1}{n}\Delta x_a}\right)_{x_a} + \frac{1}{p} \left(\frac{\Delta f}{\frac{1}{n}\Delta x_a}\right)_{x_a + \frac{1}{n}\Delta} + \dots + \frac{1}{p} \left(\frac{\Delta f}{\frac{1}{n}\Delta x_a}\right)_{x_a + \frac{p-1}{n}\Delta}. \end{aligned}$$

From this and the definition of $\varepsilon_A(x_a)$ given above it follows that

$$\left(\frac{\Delta f}{\Delta x_a}\right)_{x_a} - \left(\frac{\Delta f}{\frac{p}{n}\Delta x_a}\right)_{x_a} < \varepsilon_A(x_a)$$

in absolute value.

¹⁾ see however footnote 1 on page 90 [note 3, p. 57].

²⁾ which does not alter the fact that often discontinuities are introduced in the mathematical image as an approximation to simplify calculations, where in nature no more than a rapid transition is postulated.

Hence the oscillation $\sigma_\Delta(x_\alpha)$ of the difference quotients for increments of x_α equal to a proper fraction of Δ , is smaller than $2\varepsilon_\Delta(x_\alpha)$.

Assuming in a given point an infinite series of indefinitely decreasing increments δ , in rational proportions to Δ , the oscillations $\sigma_\delta(x_\alpha)$ will decrease indefinitely in length, whilst each consecutive oscillation is contained within the previous one; the oscillations therefore converge to a single point. This result can immediately be extended to irrational δ (because of the continuity of the functions). The difference quotient for an increment δ falls within the oscillation $\sigma_\delta(x_\alpha)$; thus the former, as well as the latter, converges to the same point. Thus there exists a limiting value for the difference quotient, viz. the derivative. Analogously, starting from the same postulate, the existence of all higher derivatives can be proved.¹⁾ (Here the basic idea was that the primitive arbitrary scales, e.g. measurement of time by means of a pendulum, measurement of length by means of a measuring rule, possess a kind of absolute truth, that is a kind of absolute equality for adjacent equal parts of the scale: these scales have in any case been constructed in a fairly intuitive manner.)²⁾

Principles of mechanics

Further³⁾ it was not assumed arbitrarily, but observed in practice a posteriori, that a large group of observed phenomena can be expressed by means of differential equations of the second order ('principle of inertia'). On the basis of this observation *force* could be introduced as a fertile auxiliary concept (the name 'force' being chosen in analogy with the influences exerted by our own bodies, which, roughly speaking, only exert an influence on the second differential quotients).

It was also observed that appropriate coefficients for bodies, called *mass*, could be introduced in such a way that very often systems could be considered as what is called *almost isolated*. This happens when one observes that the motion of the centre of gravity is approximately rectilinear and there can be shown to be an almost invariable axis of maximal momentum of the relative motion, through the centre of gravity.

¹⁾ whereby however the certainty that the function is analytic has not yet been obtained; see the remarks by PRINGSHEIM, Math. Annalen 44 (1904), p. 41–56.

²⁾ Where it was necessary to introduce different behaviour for functions in ranges of different order of magnitude (e.g. for molecular theories) it was endeavoured to retain the differentiability for both ranges of magnitude, even when it is assumed, in order to simplify calculations, that one measure is infinitely small compared with the other. In this case however one is often confronted by difficulties in the range of the larger measure. Thus, for some time it seemed difficult to retain the indefinitely continued differentiability of the potential function determined by means of DIRICHLET's principle (see HILBERT, Ueber das Dirichletsche Prinzip, Jahresber. Deutsch. Math. Ver. 8 (1899), p. 184–188 [Ges. Abhandlungen III, p. 10–14] and Math. Annalen 59 (1904), p. 161–186 [Ges. Abhandlungen III, p. 15–37]), where nevertheless the indefinitely continued differentiability is proved).

³⁾ For this and the following four paragraphs see POINCARÉ, La science et l'hypothèse, Chap. 10.

As a consequence the principle of equivalence of action and reaction was introduced in mathematical mechanics and systems were called isolated only when the motion of the centre of gravity was strictly rectilinear and uniform. And it turns out that a dynamics of rigid bodies, as well as a theoretical astronomy, can be practically controlled by mathematics built up on the notions of force and mass.

Mechanical interpretations of nature

The essential issue has been condensed into LAGRANGE's equations of motion; and it has turned out that nearly all sections of physics in which one is concerned with reversible changes can be represented by analogous equations.¹⁾

POINCARÉ²⁾ has proved that for all these phenomena an interpretation in terms of a rigid mechanism is possible. That so often such interpretations have been sought even in those fields in which another interpretation based on the existence of continuous and elastic matter seems more plausible (e.g. in the modern theory of gases and in WILLIAM THOMSON's explanation of elasticity by means of the principle of the gyrostatic spring), is probably the result of the fact that man is more familiar with the setting up of rigid constructions and mechanisms, and that he can control rigid bodies in their behaviour more easily; and that therefore the idea that nature only builds rigid mechanisms removes its mystery in so far as nature were to build things which man could not imitate building from matter; and also of the fact that in this way the considerable reliance on the invariability of the laws governing rigid bodies strengthens the illusion that 'nature can be controlled'.³⁾

Since the interpretations of LAGRANGE's equations in terms of rigid mechanics generally become very complicated and reduction to other, e.g. electrodynamical, phenomena could be made just as well as to the rigid group, it would perhaps be better to admit all groups of phenomena which can be represented by LAGRANGE's equations as having equal rights, and forgo constructing them by

¹⁾ while all other sections can be controlled by appealing to the principles of thermodynamics.

²⁾ l.c. Chap. 12.

³⁾ Or is the cause rather that a rigid body is the familiar example of an object determined by a finite number of coordinates, so that in this way, in the totality of the possibilities of a physical phenomenon, only a finite number of variables will enter and never an arbitrary function? The further consequence of this tendency would be that the remaining coordinates could only show discontinuous variations and could therefore be determined by means of integers, so that nature would only possess the order of freedom of a permutation group; everything possible in nature could be imitated by the juxtaposition in time and space of a finite number of elements in different admissible combinations (Compare here the remarks on p. 85 [p. 54, 55]). That would bring nature still closer to the material structures of man, and to the limited freedom experienced in their creation.

Of course even with this conception the usual continuous functions for the description of nature would remain in use; they would appear as approximations of large numbers with small discontinuous variations.

90

91

means of elementary phenomena in the microscopic realm, as long as such a construction does not suggest us ways to the discovery of new phenomena.

Value of the 'explanation' of phenomena

92 For the value of explanations does not lie in substituting for a wonderful phenomenon a less wonderful one, nor in the greater perspicuity with which they allow the observed sequences to be catalogued ¹⁾, but in the splitting up of observations into *essential* and *accidental* components; also in the direction indicated by the splitting up, in which the actually observed can be extended to a larger field of *possibilities*, whereby new phenomena subject to regularity can be predicted. ²⁾

In other words, an effective 'explanation' opens a field of induction; when that field is grazed, then the explanation loses its topical significance; of what remained essential a new part will be separated and considered as accidental ³⁾ so as to create a new field of induction.

93 Both the occasion and the indication for the choice of this separation are often provided by the discovery that two groups of phenomena which were previously not known to be connected and each of which was governed by its own regularity, can come under each other's influence, i.e. that they can disturb each other's isolation when brought into proximity. Then science tries to choose the essential part in the mathematical image of each in such a way that they appear as two accidental modifications of one and the same continuous region of possibilities, and that the common essential element is rendered as large as possible. In this way effective fields of induction, in which many phenomena which were previously unknown could be forecast with accuracy, have often been opened up.

Let us remark further that it can never be said afterwards that an explanation, which served its purpose in extending the region of known sequences by mean of induction, was shown to be incorrect. For, in that case, a clash with experience proves no more than that on the strength of the explanation a field of induction was opened which was *too large*. In such a case the explanation can always be saved by re-extending the essential parts in the mathematical image of the phenomena on which it was based, at the expense of the part which had been assumed as essential.

Such an explanation keeps a historical significance as the origin of a certain field of induction; but a more advanced significance is reserved for no explanation

¹⁾ As such different explanations are often equally suitable; consider e.g. the different hypotheses concerning the interaction of elements of currents, which showed themselves able to replace the hypothesis given by AMPÈRE; nor is it excluded that next to the molecular theories other equivalent theories will be developed.

²⁾ in other words in the indication of quantities in the mathematical picture of the phenomena, which can be considered to be accidental values of a variable parameter.

³⁾ He who believes in the reality of hypotheses will speak of 'going deeper into the nature of the phenomena'.

whatever. For, any new explanation which, in replacing an old one, covers a larger field of induction, will in its turn have to disappear when the limits of that field are reached; because man will always wish, and be able, to extend the totality of inductively summarized phenomena, in the realm of which he can *forecast results* and consequently intervene successfully.

Problems of space and time

We shall now touch upon the classical problems of *space* and *time* by enquiring generally in how far *objectivity* and *apriority* can be ascribed to mathematical systems. These questions, which still played an important part for RIEMANN and HELMHOLTZ, have since only interested philosophers and have been left aside by mathematicians in their researches on the foundations of their science. Only quite recently a book has been published which raises the above mentioned philosophical problems again in the light of the most recent results of mathematics. This work by B. A. W. RUSSELL, 'An essay on the Foundations of Geometry' [Cambridge 1897], has drawn general attention. It led to prolonged discussions in the Revue de Métaphysique et de Morale [1898, 1899] between POINCARÉ, COUTURAT, LECHALAS and the author himself, in which it came to light that some of the propositions in the book were untenable, but at the end of which a large part of the results was conceded to possess permanent value. COUTURAT even speaks of a *κτῆγμα εἰς ἀεί* and of no less than a perfecting of KANT's Transcendental Aesthetic.

We shall start by taking up our own stand with respect to the subject, after which we shall discuss the work of RUSSELL in greater detail by first drawing attention to a few mathematical errors in it, and then investigating in how far the purport of the book can be vindicated.

Objectivity

First, then, about *objectivity*: the *mass* of bodies is called objective, and there one thinks of its indestructibility; we have seen above however that mass is nothing but a coefficient whose introduction simplifies the mathematical image of nature, and which remains invariant for mathematical transformations representing natural phenomena. If now natural phenomena would be discovered whose representation would be simplest by assuming variable mass, then mass could only go on being called objective on the ground of its invariance for a *very important group of phenomena* in the image of nature. It must however be remembered that this image of nature would have been arbitrarily chosen, for its simplicity and suitability to be sure, but nevertheless arbitrarily; that on the one hand the image might have been constructed, though forcing the issue, in such a way that the mass remains invariant for *all* the known phenomena, but on the other hand also in such a way that it remains invariant for only very few phenomena or that it even does not appear at all.

96 One can consequently only speak of objectivity (for quantities or laws) with respect to a given mathematical image of nature and relative to a given group of phenomena. Or if one wants to speak of objectivity per se, it can mean nothing but:

either invariance for a *certain given* interpretation of all phenomena which are so far known; in which case it would be a property which we could introduce arbitrarily, albeit at the price of having to construct forced systems;

or invariance for the simplest or the most common interpretation of *all* phenomena so far known; in which case it would be a property which could disappear at any moment;

or invariance for the simplest or most common interpretation of a *very important* group of phenomena;

the last definition is the greatest mainstay, and though there remains a factor of subjective appreciation in it, one will not hesitate to call for example mass, energy and the Newtonian attraction objective in this sense, while objectivity is denied for example to temperature, magnetization and magnetic attraction.

Let us adhere to the last definition, and ask for example *in how far physical time and space are objective*. The answer must be that they possess this graduated property very completely, perhaps more completely than any other physical entity.

97 For in the first instance the fictive one-dimensional coordinate, on which the one-parameter group, being the *scientific measure of time*, is constructed, penetrates into nearly all mathematical images of nature always with the same group, which assumes the character of an unalterable invariant.

This is valid to an even greater extent for the three-dimensional Cartesian space with the six-parameter group constructed on it (the Euclidean group), which is the *physical space*; because *all* known physical phenomena can be referred to it and without it the mathematical projection of those phenomena would become extraordinarily difficult. (It is perhaps not too risky to connect this with the fact that the fictive Euclidean space is derived from the group of motions of physical rigid bodies, and that we have to rely on those bodies for all our measurements. Since it is possible to measure magnitudes only in so far as they are related to the rigid bodies, it is not surprising that the mathematical image of the group of motions of the rigid bodies continues to play such an integral part in the mathematical image of the relations between magnitudes.)

Apriority

Next the *apriority*; this can mean one of two things:

- 98 1°. Existence independent of experience.
2°. Necessary condition for the possibility of science.

If the first alternative is meant, then it follows from its intuitive construction that all mathematics is a priori, e.g. the non-Euclidean geometry as much as the Euclidean, the metrical as much as the projective.

Now let the second alternative be meant. Where scientific experience finds its origin in the application of intuitive mathematics to reality and, apart from experimental science, no other science exists barring only the properties of intuitive mathematics¹⁾, we can call *a priori* only that one thing which is common to all mathematics and is on the other hand sufficient to build up all mathematics, namely the intuition of the many-oneness, the basic intuition of mathematics.

And since in this intuition we become conscious of time as change per se, we can state:

The only a priori element in science is time. ²⁾

Coming to RUSSELL's book, we first point out the following inaccuracies:

[We refer to the first edition, 'An essay on the Foundations of Geometry', Cambridge 1897.]

99

[[4]]

1. *Space as a Zahlenmannigfaltigkeit does not presuppose free mobility.*

RUSSELL tries to prove that the axiom of RIEMANN and HELMHOLTZ, namely that space is a 'Zahlenmannigfaltigkeit', presupposes the axiom of free mobility.

From his none too brief reasonings on this point we cite those parts which are most clearly formulated:

(§ 62) '... for all the necessary adjectives of space are presupposed in any judgment of spacial quantity, and cannot, therefore, be consequences of such a judgment.' ... 'In formulating the axioms of metrical geometry, our question should be: What axioms, i.e. what adjectives of space, must be presupposed, in order that quantitative comparison of the parts of space may be possible at all?' ...

100

(§ 64) 'But if 'measurement *consists* in a superposition of the magnitudes compared', does it not follow immediately that measurement is logically possible *only* where such superposition leaves the magnitudes unchanged? And therefore that measurement, as above defined, involves, as an *a priori* condition, that magnitudes are unchanged by motion?' ...

(§ 144) 'We require, then, at the very outset, some criterion of spatial equality; without such a criterion metrical Geometry would become wholly impossible. It might appear, at first sight, as though this need not be an axiom, but might be a

¹⁾ Strictly speaking the construction of intuitive mathematics in itself is an *action* and not a *science*; it only becomes a science, i.e. a totality of causal sequences, repeatable in time, in a mathematics of the second order, which consists of the *mathematical consideration of mathematics* or of the *language of mathematics*; only there one meets with causal connections in the way in which mathematical systems on the one hand, mathematical symbols, words or ideas on the other hand, succeed one another. But there, as in the case of theoretical logic, we are concerned with an *application of mathematics*, that is with an *experimental science*. Cf. in this connection the developments of the third chapter.

²⁾ Of course we mean here *intuitive time* which must be distinguished from *scientific time*. By means of experience and very much *a posteriori* it appears that scientific time can suitably be introduced for the cataloguing of phenomena, as a one-dimensional coordinate having a one-parameter group.

mere definition. In part this is true, but not wholly.' ... 'It follows that the application of the conception of magnitude to figures in space involves the following axiom: *Spatial magnitudes can be moved from place to place without distortion.*' ... 'that metrical Geometry, if it refused this axiom, would be unable, without a logical absurdity, to establish the notion of spatial magnitude at all' ...

(§ 153) 'We spoke above of the Geometry on an egg, where Free Mobility does not hold. What, I may be asked, is there about a thoroughly non-congruent Geometry, more impossible than this Geometry on the egg? The answer is obvious. The Geometry of non-congruent surfaces is *only* possible by the use of infinitesimals, and in the infinitesimal all surfaces become plane. ... *If we had not our Euclidean measure, which could be moved without distortion, we should have no method of comparing small arcs in different places.*' ... [[italics of Brouwer]]

This last sentence is wide open to disproof. For we can construct a Cartesian space in which arbitrary systems of surfaces are taken as coordinate-planes, while in each coordinate an arbitrary one-parameter group is fixed as basis for the determination of measurement; then we can define an arbitrary function of the infinitesimal increments of coordinates as an element of arc, and as distance between two points, their geodesic distance. To be able to compare quantities in different parts of space, we need consequently not assume the possibility of displacement for three-dimensional bodies, but only for one-dimensional threads. And here we do not even necessarily obtain in the infinitesimally small a geometry with free mobility, witness for example the Minkowski geometry (see Chapter I, p. 59 [[p. 43]]).

2. A space constant variable in time is easily conceivable.

In § 100 the author states that it is *inconceivable* that the space constant could vary with the time.

'This would involve a causal connection between space and other things, which seems hardly conceivable, and which, if regarded as possible, must surely destroy Geometry, since Geometry depends throughout on the irrelevance of Causation. Moreover in all operations of measurement, some time is spent; unless we knew that space is unchanging throughout the operation, it is hard to see how our results could be trustworthy, and how, consequently, a change in the parameter could be discovered.' [[The words 'unless ... operation' are left out in the French translation.]]

If this means that a space with a variable constant is inconceivable, we can reply: Imagine an expanding sphere. All rigid bodies on it are deformed in a definite manner; with respect to each other these bodies will also be deformed, but at every moment the deformation is invariant with respect to displacements. Consequently the rigid group remains a rigid group but the space constant for the displacement group is changed. If it means that the empirical space-constant can never vary, then it is in fact true that we can never discover a variance, because all we do is to catalogue our empirical phenomena in a Euclidean space

created by ourselves. We can maintain the Euclidean space independent of the phenomena; but we can catalogue just as well in a space which has a different curvature at different moments.

103

The observations with our astronomical instruments are catalogued most simply if we assume a continuation of the usual terrestrial Euclidean space in celestial space and in it a rectilinear prolongation of light rays, which strike our telescopes. It is, however, not impossible that observations of stars with very small parallax could be catalogued more simply as follows: instead of assuming for the bundle of directions of light rays centred at the observation-point the relation at very great distances between the measure *along* the rays and that *between* the rays in the form

$$ds_{\text{perpendicular to } r} = r d\varphi,$$

we assume it in the form

$$= \frac{1}{\alpha} \sin \alpha r \cdot d\varphi$$

or

$$= \frac{1}{\alpha} \operatorname{sh} \alpha r \cdot d\varphi$$

(where α is so small, that only for very great distances a perceptible difference with the formula $= r d\varphi$ would occur). We could even assume other relations which would not give the same constant in every point of space.

Nor can we give an a priori reason why this space constant, and therefore this principal group of motions could not vary, for example under mutual influences of different systems of celestial bodies. One could even state:

At the moment when the suitability of the constant α were to be discovered, the space would suddenly turn from Euclidean into non-Euclidean; in the same way as POINCARÉ put it that the earth only rotated from the time that COPERNICUS stated it.

104

3. *There are no circular reasonings in the construction of geometry.*

In § 108 it is stated: 'All geometrical reasoning is, in the last resort, circular: if we start by assuming points, they can only be defined by the lines or planes which relate them: and if we start by assuming lines or planes, they can only be defined by the points through which they pass.' The *construction* of the Euclidean geometry already proves the inaccuracy of this statement.

4. *For experience a higher-dimensional continuum is not a necessary condition.*

In § 186–192 the following proposition is discussed: 'The existence of diverse but interrelated things would be unknowable, if there were not, in sense-perception, some form of externality.' The essential point of reasoning amounts to pointing out that what we have called the basic intuition of mathematics, is indispensable for every intellectual function. In § 191 in particular the author tries to prove that *time alone* is not sufficient, on the argument that in that case no

objects could be perceived. We answer: The simplest causal sequences which man can perceive have in reality for their mathematical substratum only *time* as a one-dimensional intuitive continuum; it does not matter that in them no other objects, that is invariants, than time itself, appear. Objects only appear in portions of experience with a more complex mathematical substratum, in the same way as invariants enter only in mathematical systems of greater complexity.

In connexion with the above mentioned proposition, it is maintained in § 135 that the form of externality, which should, next to time, be admitted as a necessary condition for experience and which will have to appear as empirical space, must of necessity have more than one dimension. 'For in a one-dimensional form, the various contents may be arranged in a series, and cannot, without interpenetration, change the order of contents in the series. But interpenetration is impossible, ... A form of one dimension, therefore, could not, by itself, allow that change of the relations of externality by which alone a varied world of interrelated things can be brought into consciousness.' To which we answer again that such a world of *objects (things)* is not a necessary condition for experience; that empirical space is an arbitrary creation to enable us, all the same, with the aid of mathematical induction to bring *different* causal sequences (of results of measurements) together *under one point of view*; and that the creation of such an idealizing summary of real experiences as parts of a fictive totality of possible experiences, does not entail the result that invariants which merit the name of objects will enter into real experience.

The only remark which can be made in this connection in the direction of RUSSELL's line of thought is that, as soon as mathematical induction enters in the mathematical image of experience as a means of taking together *different* sequences, the basic intuition of mathematics will enter there twice independently and in different ways; that will lead to a higher-dimensional continuum if we understand it as continuous in both cases; but the mathematical *existence* of a higher-dimensional continuum is independent of experience, and its applicability to experience is a posteriori.

A higher-dimensional form of externality per se is a fortiori not necessary for experience.

If the introduction of a higher-dimensional continuum is therefore not necessary a priori, it is even less so for a higher-dimensional continuum with such relations between the elements that it can be considered as a form of externality per se, as in the case of the one-dimensional intuitive continuum.

And only on the basis of these incorrect opinions RUSSELL (§ 129–139) develops the following properties of empirical space as being necessary for experience, deducing them ¹⁾ from the conception of the *form of externality per se*, realized in a higher-dimensional continuum:

¹⁾ These deductions are themselves also incorrect; see POINCARÉ, Des fondements de la géométrie § 4, Revue de Métaphysique et de Morale, 1899, p. 251–279.

I. 'Space is continuous and infinitely divisible; the result of infinite division, the zero of extension, is called a *point*. All points are qualitatively similar and can only be distinguished by the fact that they are mutually external to one another.'

II. 'Any two points determine a unique figure, called a straight line. Two straight lines, like two points, are qualitatively similar and can only be distinguished by the fact that they are mutually external to one another.'

III. 'Any three points which are not collinear determine a unique figure, the plane, and four points not lying in one plane determine a figure of three dimensions. This progression can as far as one can judge *a priori* be prolonged up to five or n points without excluding in any way the possibility of a projective Geometry. But projective Geometry demands, through its axioms, that this progression stops at a positive integral number of points, after which every new point must be contained in the figure determined by those which are already given. If this progression stops at $(n+1)$ points, it is said that the space has n dimensions.'

5. Projective geometry is not necessary for experience.

Further it is a mathematical blunder to describe the above mentioned axioms as a complete characterization of projective geometry. ¹⁾ For the core of the projective axioms, namely the properties that the p space determined by $(p+1)$ points contains every straight line having 2 points in common with it, and that a straight line and an $n-1$ space must intersect, are missing, and cannot be deduced from RUSSELL's axioms. In fact, in a Cartesian space we can construct an arbitrary number of systems of curves, surfaces etc., which satisfy the projective axioms. Thereupon we can define the relation between 2 points as the straight line of any one system, and the relation between 3 points as a plane from another system; in this way the axioms of RUSSELL are satisfied, but not the projective axioms.

The creation of the projective system is not only unnecessary, but also anything but primitive or simply determined, as appears clearly from the developments of HAMEL which I summarized in Chapter I, p. 57 [p. 41] seq., and which show how the linear systems can be determined in many different ways. These ways can again be multiplied indefinitely by arbitrary uniform point transformations; for in this way the intrinsic properties of the straight lines are retained in the geodesic curves into which the lines are transformed, so that they will always allow a transformation group, isomorphic with the projective group.

6. Free mobility is not necessary for experience.

The author also tries to deduce the axiom of free mobility from the concept of *form of externality per se* (in particular § 143–157).

(§ 62) 'The conditions of measurement themselves, though not results of any conception of magnitude, will be *a priori*, if it can be shown that, without them, experience of externality would be impossible.'

¹⁾ See POINCARÉ, I.c. § 3.

(§ 145) 'Space must, since it is a form of externality, allow only of relative, not of absolute, position, and must be completely homogeneous throughout.'

We shall not discuss the reasonings relative to this question, which form perhaps the weakest part of the work; it is, of course, clear that what was proved to be incorrect for the projective group, can a fortiori not hold for the more special groups of the Euclidean and of the non-Euclidean motions.

We only remark that COUTURAT (criticism of RUSSELL, *Revue de Métaphysique et de Morale* 6 (1898), p. 372 seq.) infers a perfectly correct consequence from the above mentioned thesis. RUSSELL states in § 145: 'Space must, since it is a form of externality, allow only of relative, not of absolute, position, and must be completely homogeneous throughout.' In other words (§ 144): 'Shapes do not in any way depend upon absolute position in space.'

110 COUTURAT then points out that where a reason can be found to deny any influence of absolute *orientation*, as much reason can be given to deny the influence of absolute *magnitude*, since at most relative quantities, and in no case absolute quantities, can occur in the concept of the *form of externality per se*. And from this he deduces that not only the axiom of free mobility, but also the Euclidean axiom of parallel lines and consequently the whole Euclidean geometry must be considered to be a priori.

COUTURAT is correct in his criticism of RUSSELL, but we repeat: The form of externality is only a priori in *one* dimension, and there neither absolute, nor even relative, quantities occur in it. Relative quantities only appear after a one-parameter continuous and uniform group has been constructed as an arbitrary mathematical construction on the one-dimensional continuum (in such a construction the *element*, i.e. the basic intuition of the form of externality, is unalterable and a priori, but what remains arbitrary is the way of linking together the repeated applications of the basic intuition to that which has been previously constructed or parts thereof).

7. *The projective distance does not presuppose 'ordinary' distance.*

In § 37 the author tries to retain *distances* on a straight line as primitive conceptions. It is said that KLEIN's projective introduction of distances by means of the quadrilateral construction arbitrarily defines *something other* than distance, and that this definition is impossible without a previous conception of what RUSSELL calls 'distance in the ordinary sense'.

111

As an illustration we cite (§ 37): 'If A, B, C be three different points on a line, there must be *some* difference between the relation of A to B and of A to C, for otherwise, owing to the qualitative identity of all points, B and C could not be distinguished. But such a difference involves a relation, between A and B, which is independent of other points on the line; for unless we have such a relation, the other points cannot be distinguished as different. Before we can distinguish the two fixed points, therefore, from which the projective definition starts, we must already suppose some relation, between any two points on our line, in which they

are independent of other points; and this relation is distance in the ordinary sense. ...Distance, in the ordinary sense, remains a relation between *two* points, not between four.'

Here he loses sight of the fact that one can very well conceive a continuum without being able to compare 'magnitudes' thereon. This becomes possible only after preference is given to an arbitrary one-parameter group. (As a matter of fact RUSSELL himself touches upon this difference between 'intensive' and 'extensive' magnitudes in § 178 and emphasizes it later in his 'Principles of Mathematics' [p. 182].) Furthermore it is incorrect to call the linear distance a relation between *two* points, it can only occur in relations between two distances, hence in relations between at least *three* points. This insight is also obtained from the one-parameter group with which every scale, and hence every determination of distance, is equivalent.

112

8. Lie's results

In § 45 the results of LIE are reproduced incorrectly. It is stated: '*In two dimensions*, if free mobility is to hold *universally*, there are no groups satisfying HELMHOLTZ's first three axioms, except those which give the ordinary Euclidean and non-Euclidean motions; but if it is only to hold *within a certain region*, there is also a possible group in which the curve described by any point in a rotation is not closed, but an equiangular spiral. To exclude this possibility, HELMHOLTZ's axiom of monodromy is required.'

But this is not so. If free mobility is to apply to the whole space,¹) then all Euclidean and non-Euclidean motions can no longer be taken into consideration, but only the non-Euclidean elliptic group (see LIE, 'Ueber die Grundlagen der Geometrie', Leipziger Berichte 1890, p. 289).

113

Consequently the postulate of free mobility *over the whole space* cannot serve to retain the Euclidean and non-Euclidean motions and to exclude the spiral group. One must postulate either that pseudocircles do not contain their centre (not even as a limiting point) or HELMHOLTZ's postulate of monodromy.

Further it is stated:

'If, in three dimensions, free mobility, within the specified region, holds only of every point *of general position*, while the points of a certain line, when one point is fixed, are only able to move on that line, not on the surface; when this is the case, other groups are possible, and can only be excluded by HELMHOLTZ's fourth axiom.'

LIE's result however amounts to this: 'and cannot be excluded even by HELMHOLTZ's fourth axiom'.

¹) LIE means the complete projective space, for which the Euclidean and hyperbolic groups have their fundamental conic as unattainable points, and such an unattainable point does not possess the property that if it is fixed together with an arbitrary line-element through it, the whole plane will then be fixed.

KANT's *point of view*

We now come to the main purport of the work, which was intended to rectify KANT's point of view on apriority in the experience and bring it up to date.

KANT defends the following thesis on space:

114 The perception of an external world by means of a three-dimensional Euclidean space is an invariable attribute of the human intellect; another perception of an external world for the same human beings is a contradictory assumption.

KANT proves his thesis as follows: ¹⁾

Of empirical space we notice two things:

1°. We obtain no external experiences barring those placed in empirical space, and cannot imagine those experiences apart from empirical space (i.e. under (1) and (2));

2°. For empirical space the three-dimensional Euclidean geometry is valid (i.e. under (3));

from which it follows that the three-dimensional Euclidean geometry is a necessary condition for all external experiences and the only possible receptacle for the conception of an external world so that the properties of Euclidean geometry must be called *synthetic judgments a priori* for all external experience.

115 Both premisses serve to demonstrate in a certain sense (which is stronger than that meant by us on page 96 [p. 60]) the *objectivity* in the first instance of empirical space per se, without which no external experience is said to be accessible to thought (this probably only means a three-dimensional Cartesian space), and secondly of the group of Euclidean motions constructed thereon. But it can immediately be objected that we *obtain* our experiences apart from all mathematics, hence apart from any space conception; mathematical classifications of groups of experiences, hence also the creation of a space conception, are free actions of the intellect, and we can arbitrarily refer our experiences to this catalogization, or undergo them unmathematically.

Consequently the addition to the first premiss, that we cannot imagine our known external experiences apart from our space conception, is definitely false. With it the conclusion basing the *apriority* of three-dimensional Euclidean geometry mainly on this addition, must also be rejected.

116 But even accepting KANT's premisses, we can advance the following argument against the conclusion: cannot the human intellect be organized as well in such a way that it can place the conception of an external world in other receptacles, without this being realized *in practice*; for instance because little effect can be obtained from it and consequently our capacity for it receives very little practice? Empirical rigid bodies are the only ones to which the human measuring instinct can apply itself; that explains why the group of motions of these rigid bodies gradually became the schema of human understanding on results of measurements.

¹⁾ Kritik der reinen Vernunft, ed. Kehrbach, p. 50–52 [B 50–52].

But though our virtuosity in relating the phenomena of the world of experience to this schema is considerable, this does not exclude us from practising not only to construct other schemas, but also to relate our experiences to them. The human intellect undergoing external experiences can therefore, if that is the case, disengage itself very well from three-dimensional Euclidean geometry.

Russell's point of view

This last point is also upheld by RUSSELL, but he wants to retain as necessary properties of the receptacle the properties of projective geometry and the axiom of free mobility, so that only a choice remains between the Euclidean, elliptic and hyperbolic geometries of a certain number of dimensions.

Furthermore he is of the opinion, that, although the human intellect is organized for these different geometries, experience teaches us that only the three-dimensional Euclidean geometry can be serviceable for this accidentally given reality, for which it would be true to a high degree of approximation (see e.g. § 209).

We have pointed out above how RUSSELL deduces the first of these two propositions:

for projective geometry out of an incorrect claim that the mathematical substratum of experience possesses more than one dimension, and the incorrect extension to space constructed from more dimensions of the conditions which a *form of externality per se* must satisfy;

for metrical geometry out of the arbitrary introduction of *measurability* (which can only be done by the construction of a group, as indicated here in the first chapter, and which is not a necessary condition for experience), and further again out of the incorrect generalization to higher-dimensional spaces of postulates which are only justified for one dimension.

As for the second proposition, *there does not exist a definite empirical space*: we can catalogue all phenomena in every space, with any number of dimensions, and curved as grotesquely as we wish, hence also without free mobility. Empirical *science* is bound up with mathematics, but experience can never compel us to the choice of one *definite* mathematical system.

The three-dimensional Euclidean geometry is a six-parameter group, in which the motions of empirical rigid bodies in our immediate neighbourhood can be represented with a very high degree of approximation. Further a substratum of the phenomena of nature, which are studied by man, can often be easily submitted to a law in the group of motions of empirical rigid bodies (the most appropriate method for this being in practice the construction, by means of the said group, of empirical curves which are called *straight lines*, and the construction on these straight lines of scales, called *scales of distances*). This substratum can be used as a means to control these phenomena for many purposes, and has enabled man to construct serviceable measuring implements for technology and physics based on empirical rigid bodies. In this way Euclidean geometry, that is the

117

118

Euclidean mathematical *group* became the basis for all understanding of man on all phenomena of his world of experience.

Through its constant use Euclidean geometry has become a most generally serviceable part of mathematics, but it can very well be imagined that with the same organization of human intellect another mathematical construction would have become as popular.

Summary of the relation between mathematics and experience

Summing up our point of view with regard to both main points of KANT's Transcendental Aesthetic:

a). *With respect to what is inseparable from external experience*: Not only does, as was stated on p. 98 [p. 60], mathematics exist independently of all experience, but all experience is also independent of all mathematics. Human experience is not passively subjected to any single mathematical system; not even to the coordinate of time, nor even to the time-continuum devoid of measure.

b). *With respect to the necessary occurrence in the mathematical receptacle of experience*: The necessity exists only for the primitive intuition of mathematics, because the mathematical receptacle of experience is not subject to any restriction barring mathematics itself. Mathematics develops out of its basic intuition in a self-multiplication guided by an entirely free choice. The only synthetic judgments a priori generally, are therefore those which are obtained as possibilities of mathematical constructions by virtue of the basic intuition of time, or of many-one-ness, in other words which are obtained as possibilities of sets of points on the continuum.¹⁾ Consequently there can be mentioned as such judgments:

1°. the very possibility of mathematical synthesis, of thinking many-one-ness, and of the repetition thereof in a new many-one-ness.

2°. the possibility of intercalation (namely that one can consider as a new element not only the *totality* of two already compounded, but also that which *binds* them: that which is not the totality and not an element).

3°. the possibility of infinite continuation (axiom of complete induction).

Experience a posteriori can teach us nothing about the necessity of the occurrence of definite mathematical systems in experimental *science*.

The following summary gives a ready comparison between the points of view of KANT and RUSSELL and that developed here:

¹⁾ One must however not try to base mathematics or experience on these judgments: the latter are the result of the *mathematical contemplation* of the basic intuition and hence presuppose the basic intuition in the contemplation as well as in that which is contemplated; they belong to what we shall call in the next chapter *mathematics of the second order*.

	In KANT's Transcendental Aesthetic	In RUSSELL's Foundations of Geometry	In this work
Inseparably linked up with external experience is:	the three-dimensional Euclidean space and the time devoid of measure.	the three-dimensional Euclidean space and the measurable time-coordinate.	nothing.
Necessarily appearing in the mathematical receptacle of experience is:			
a) by virtue of the organization of the human intellect:	the three-dimensional Euclidean space and the time devoid of measure.	the projective space, free mobility in space and the measurable time- coordinate.	the basic intuition of mathematics or the intuition of time.
b) by virtue of experience:	nothing.	the three- dimensionality of space and the parallel-axiom of Euclid.	nothing.

III. MATHEMATICS AND LOGIC

125

Mathematics is independent of logic.

We want to show that mathematics is independent of the so-called *logical laws* (laws of reasoning or of human thought). This seems paradoxical, for usually mathematics is expressed, orally or in writing, in the form of argumentation, deduction of properties, by means of a chain of syllogisms. But the conceptions which are evoked by the words used in such an explanation, consist in the following: Where mathematical objects are given by their relations to the basic or complex parts of a mathematical structure ¹⁾, we transform these given relations by a sequence of tautologies ²⁾ and thus gradually proceed to the relations of the object to other components of the structure.

126

The proofs which we gave in Chapter I for the very first theorems of mathematics, taught us to read these theorems as tautologies. The fact that in more complicated cases a theorem is not immediately clear, but is only understood after a chain of tautologies, proves merely that we build our structures too complicated to be comprehended in one view.

In one particular case the chain of syllogisms is of a somewhat different kind, which seems to come nearer to the usual logical figures and which actually seems to presuppose the hypothetical judgment from logic. This occurs when a structure is defined by some relation in another structure, while it is not immediately clear how to effect its construction. Here it seems that the construction is *supposed* to be effected, and that starting from this hypothesis a chain of hypothetical judgments is deduced. ³⁾ But this is no more than apparent; what actually happens is the following: one starts by setting up a structure which fulfills part of the required relations, thereupon one tries to deduce from these relations, by means of tautologies, other relations, in such a way that these new relations, combined with those that have not yet been used, yield a system of conditions, suitable as a starting-point for the construction of the required structure. Only by this construction will it be proved that the original conditions can be fulfilled.

127

'But', the logician will retort, 'it might have happened that in the course of these reasonings a contradiction turned up between the newly deduced relations

¹⁾ this means that the object in question is built in connection with the components to which it is said to be related.

²⁾ i.e. by fixing one's attention on different substructures of the mathematical system.

³⁾ Consider for instance the unicity proofs by HILBERT and LIE for transformation groups with given properties, or even the usual elementary construction problems in geometry, like the construction of a common harmonic pair to two given pairs of points, or the problems of APOLLONIUS.

and those that had been kept in store. This contradiction, to be sure, will be observed as a logical figure, and this observation will be based upon the principium contradictionis.' To this we can reply: 'The words of your mathematical demonstration merely accompany a mathematical *construction* that is effected without words. At the point where you enounce the contradiction, I simply perceive that the construction no longer *goes*, that the required structure cannot be imbedded in the given basic structure. And when I make this observation, I do not think of a principium contradictionis.

Logic depends upon mathematics.

While thus mathematics is independent of logic, logic does depend upon mathematics: in the first place *intuitive logical reasoning* is that special kind of mathematical reasoning which remains if, considering mathematical structures, one restricts oneself to relations of whole and part; the mathematical structures themselves are in no respect especially elementary, so they do not justify any priority of logical reasoning over ordinary mathematical reasoning. It could be put forward: The *successor* relation, which is essential in mathematics proper, does *not yet* occur in the mathematics of logical reasoning. Then the reply must be: It is true that this relation *no longer* occurs *explicitly*, but here as well as in all mathematics it is presupposed; for it accompanies any mathematical construction, albeit that when the construction is ready, it is no longer immediately apparent for some relations between the elements.

128

People try by means of sounds and symbols to originate in other people copies of mathematical constructions and reasonings which they have made themselves; by the same means they try to aid their own memory. In this way the *mathematical language* comes into being, and as its special case the *language of logical reasoning*.¹⁾

With which mathematical notions a spoken or written symbol will be made to correspond, this choice will take into account as economically as possible the most common mathematical systems and methods of reasoning; therefore it will in general differ according to the milieu. In particular, the answer to the question which domains of mathematics will be accompanied by a language not only among professional mathematicians, but also in daily life, will depend for every nation anew upon the question, which domains of mathematics have found most applications to the guidance of action or as a means of understanding about action.

129

Therefore it is easily conceivable that, given the same organization of the

¹⁾ Even in domains of mathematics where no relations of whole and part enter, *the relations which were in the mind are often transformed into relations of whole and part*, when they must be communicated verbally to other people; hereby the usual language of mathematics in general is imbued with that of logical reasoning. However, this fact is due only to the centuries-old tradition of logical terms in language, in connection with its limited vocabulary.

human intellect and consequently the same mathematics, a different language would have been formed, into which the language of logical reasoning, well known to us, would not fit. Probably there are still peoples, living isolated from our culture, for which this is actually the case. And no more is it excluded that in a later stage of development the logical reasonings will lose their present position in the languages of the cultured peoples.

Man, inclined to take a mathematical view of everything, has also applied this bias to mathematical language, and in former centuries exclusively to the language of logical reasonings: the science arising from this activity is *theoretical logic*.

It is only in the last twenty years (though the earliest traces go back to LEIBNITZ) that people have started looking in the same way at *mathematical language in general*: this is the content of *logistic*, insofar as it is studied without overrating its value.¹⁾

We infer that theoretical logic as well as logistic are *empirical sciences* and that they *apply* mathematics; consequently they can yield no information whatsoever on the organization of the human intellect; there would be better reasons to reckon them under *ethnography* than under *psychology*.

And the language of logical reasonings is no more an *application of theoretical logic* (for that matter, if this were the case, of which science would the language of theoretical logic itself be an application?) than the human body is an application of anatomy.

Let us, by way of illustration, consider the classical syllogism:

All men are mortal.

Socrates is a man.

ergo: Socrates is mortal.

The thoughts accompanied by these words are the following:

We start by projecting in the world of perception a mathematical system, namely a finite set of 'subjects', each of which is connected with some (none or one or more) elements of another set whose elements are called 'predicates'. It turns out that in the human intellect a part of the world of perception can approximately be projected on such a system.

Why, in such a system it is a *mathematical* tautology that, if the elements bearing the predicate 'man' are part of those bearing the predicate 'mortal', that then the element 'Socrates' from the first set also belongs to the second set. We have here one of the very simplest forms of mathematical reasoning, namely passing by a tautology from one relation to another.

However, looking at the *words* that accompany this primitive form of mathematics, we notice in them a surprising mechanism with a regularity which is not clear a priori. That is to say, it is possible to project on these words a new

¹⁾ see p. 159 seq. [p. 88 seq.]. The systems of logistic developed until now consider a mathematical language which uses excessively the words of theoretical logic, and which sometimes, where excessive use leads to illicit use, gave rise to error.

simple mathematical system; speaking about this system we explain the *theory of the syllogism*. But the mathematical systems which function in this theory are of a very simple kind, and therefore no logic is required to understand them.

While in the syllogism a mathematical element could be discerned, the proposition:

A function is either differentiable or not differentiable
says *nothing*; it expresses the same as the following:

If a function is not differentiable, then it is not differentiable.

But the logician, looking at the *words* of the former sentence, and discovering a regularity in the combination of words in this and in similar sentences, here again projects a mathematical system, and he calls such a sentence an *application of the tertium non datur*.

Further we emphasize that the syllogism and the other logical principles may be reckoned to hold for the language of logical reasonings on finite sets, on denumerably infinite sets, on domains in continua, but in any case exclusively on mathematically constructed systems; the conviction that we may rely on that applicability, is based on the *certainty* that mathematically constructed systems are under discussion.

On the basis of linguistic images which accompany basic mathematical truths in actual mathematical structures, it is sometimes possible to build up *linguistic* structures, sequences of sentences, proceeding according to the logical laws. If it turns out that such a structure can never produce the linguistic form of a contradiction, then all the same it belongs to mathematics only in its quality of a linguistic structure, and it has nothing to do with mathematics outside of it, such as ordinary arithmetic or geometry.

So the idea that by means of such linguistic structures we can obtain knowledge of mathematics apart from that which can be constructed by direct intuition, is mistaken. And more so is the idea that we can lay in *this* way the *foundations* of mathematics, in other words, that we can assure in this way the reliability of the mathematical theorems.

We shall now, on the basis of the considerations given above, discuss successively in more detail:

- 1°. The foundation of mathematics on axioms.
- 2°. CANTOR's theory of transfinite numbers.
- 3°. The logistic of PEANO-RUSSELL.
- 4°. The logical foundations of mathematics after HILBERT.

Ad 1°

EUCLID *improved*

Here the classical example is the geometry of EUCLID. It has been convincingly demonstrated by recent investigations of PASCH, SCHUR, HILBERT, PEANO, PIERI

132
[[2]]

133

and others, that as a logical linguistic structure it is imperfect, namely that here and there tacitly axioms are introduced, but these mathematicians had little trouble to improve it in this respect. However they, especially HILBERT, have occupied themselves by building linguistic structures for *pathological geometries*. The purpose of these constructions is to show, which properties (i.e. sentences, expressing geometrical properties in Euclidean geometry) can, and which cannot, be preserved if some of the axioms are dropped (following the tracks of LOBACHEWSKI, who investigated what remains of EUCLID's logical structure if the axiom of parallels is dropped ¹⁾). In particular they aimed at minimizing the necessary axioms for each of the logical structures obtained in this way. Thus HILBERT has shown for most of the axioms put forward in his Festschrift [Grundlagen der Geometrie] that the geometry loses part of its properties if the axiom is left out. ²⁾

We must remark however that EUCLID cannot be blamed for incompleteness of his axioms if he conceived his mathematical structure of Euclidean geometry as being already *finished* (in the form of a Cartesian space with a group of motions), and if his reasonings serve no other purpose than to accompany the transition by a chain of tautologies from clearly perceived relations (i.e. substructures) to new relations which are not immediately perceived; in other words, if they accompany the exploration of a structure built by himself. In this case his work belongs to pure mathematics, and the fact that he does not introduce coordinates in order to operate with them, is no more than a flaw in his method.

Of course it is also possible that EUCLID has not seen it this way and that he lapsed into the mistake which so many people made, thinking that they could reason logically about other subjects than mathematical structures built by themselves, and overlooking the fact that, wheresoever in logic the word *all* or *every* is used, this word, in order to make sense, tacitly involves the restriction: *insofar as belonging to a mathematical structure which is supposed to be constructed beforehand*. ³⁾

¹⁾ Though it is not difficult to deduce from LOBACHEWSKI's calculations a *proof of existence*; it is not impossible that he himself meant it that way; see for instance Pangeometrie, Kazan 1856, § 8 [Gesammelte geometrische Abhandlungen, Kazan 1883–1886, p. 617–680; see also Géométrie imaginaire, J. reine angew. Math. 17 (1837)].

²⁾ Still, even if he had proved this for each of his axioms – he does not investigate the 'Axiome der Verknüpfung' nor the 'Axiome der Anordnung'; he motivates this by the hazy reason that they 'bei unserer Darstellung den übrigen Axiomen zu Grunde liegen' – even then the proof of minimality would not be complete. For any axiom in which the word *every* occurs, can be split up, already in the axiom for *every except one* and that for *that last one*, and in each case it ought to be shown that the second part does not follow from the first; possibly this can be done, but certainly not quite simply, and in this respect HILBERT has left his research unfinished.

³⁾ Besides this delusion on the *freedom* of logic stands as an analogous overrating the idea of ARISTOTLE and the scholastics – which still influenced SPINOZA strongly and KANT somewhat less, and from which philosophy seems not to have emancipated before the 19th century – namely that logic would be able to uncover secrets of nature which are not clear a priori. In reality the

However this may be, posterity has in most cases conceived EUCLID's work as such a logical structure. LOBATCHEWSKI (perhaps) and BOLYAI (for certain) also built up logical structures without bothering about mathematical systems which they might accompany.

RIEMANN

RIEMANN was the first to show the right way for research on the foundations of geometry by starting from the idea that space is a *Zahlenmannigfaltigkeit*, thus a system built up by ourselves. However, he emphasizes that this is an arbitrary hypothesis, and he does not say that in *any* case we must choose a mathematical system at the basis, and that we choose the *Zahlenmannigfaltigkeit* for reasons of efficiency.

HILBERT c.s.

So it can be understood that PASCH, HILBERT and others, because they considered his hypothesis as arbitrary, resumed the logical foundation of geometry, trying to better EUCLID by constructing, as we have explained above, linguistic structures¹⁾ developed from axioms solely by means of the formal syllogism and the other logical principles. They introduce intuitive mathematics in the sphere of their considerations only for proofs of consistency (indicating a system with the property that a certain set of axioms, and consequently all the theorems deduced from it, can be held to express properties of that system²⁾) and for *proofs of*

137

conclusions reached by this method do not hold for nature itself, but only for the mathematical system which has been arbitrarily projected on nature (and only part of which covers what has been directly perceived, while the rest has been added by induction). One should verify for every conclusion anew (and every verification ought to be completed by mathematical induction) that it is also true for nature (i.e. that it is efficient as a guide for human action). Such a verification is necessary even if the premisses are completely true, in the same way as every new consequence of a physical hypothesis ought to be expressly checked, no matter how useful the hypothesis has proved so far.

Moreover this verification can lead to different results for different persons, because they check the words of the conclusion with different mathematical systems which they mentally connect with these words. It is also possible that, lacking such mathematical systems, verification is impossible for the time being and that it must wait for further experience, i.e. for the building of new mathematical systems.

¹⁾ HILBERT even declares explicitly that in connection with words like 'Punkt', 'Gerade', 'zwischen' etc. he does not think of a mathematical interpretation.

²⁾ It is clear that indicating a mathematical system such that the axioms *might* accompany properties of that system, suffices to prove that never two contradictory theorems can be deduced from the axioms, for contradictory theorems cannot hold for a mathematical structure. For that matter, the fact that mathematical systems were put forward to establish the existence of logical systems, shows that it was still felt that the mathematical system itself needed no further existence proof besides its intuitive construction. That HILBERT lost this conviction later, appears

from his note 'Über den Zahlbegriff' (Jahresberichte Deutsch. Math. Ver. 8 (1899), p. 180–184 [Anhang VI in 'Grundlagen der Geometrie']), where he considers the number systems which he had introduced to prove the existence of his geometries, as being defined in their turn by a purely axiomatic method; therefore the consistency of these number systems must again be proved without recurring to intuition. But he ought to remember that, giving this proof, he intuitively constructs a mathematical system (the system of the theorems deduced from the axioms) following logical laws, and that consistency enters in his proof as a mathematical property of that mathematical system, which can only be proved by methods based on intuition. But he is not willing to admit intuitive mathematics, so he will be forced to look at the latter proof again as a linguistic structure, as a reasoning based on axioms (namely axioms on the 'Verknüpfung' of the elements which are the theorems of the system, among which occurs *certainly* the axiom of complete induction); again it will be his duty to prove that these axioms are consistent. But 1° he is now no further forward than at the preceding stage, 2° the consistency of the axioms does not involve the existence of the corresponding mathematical system, 3° even if the mathematical system of reasonings exists, this does not entail that it is *alive*, i.e. that it accompanies a sequence of thoughts, and even if the latter is the case, this sequence of thoughts need not be a *mathematical* development, so it need not be convincing.

We shall see later on how HILBERT tried to escape from this difficulty and in how far he succeeded.

Let us recall in this connection DEDEKIND's famous monograph 'Was sind und was sollen die Zahlen?', in which he aims at proving logically the arithmetic of whole numbers, starting from the most primitive notions. For this purpose he constructs a logical system (i.e. a mathematical system of words), the axioms of which are the linguistic images of the connections between the basic notions (*whole and part, correspondence between elements, mapping of systems, etc.*) and which further is *finitely* constructed following the logical laws (thus without using complete induction, i.e. the mathematical intuition of 'and so on'). In order to have mathematical significance, this system ought to be completed by a mathematical existence proof. But in order to give that, we shall certainly be forced to use the intuition 'and so on', and then we see at once that we can obtain all the arithmetical theorems much more easily than by DEDEKIND's contrived system; accordingly DEDEKIND does not give the existence proof. He does give in § 66 a proof for: 'Es gibt unendliche Systeme', but 1° a proof is needed for: 'Es gibt einfach unendliche Systeme', what is more, and 2° his proof, which introduces 'meine Gedankenwelt' is false, for 'meine Gedankenwelt' cannot be viewed mathematically, so it is not certain that with respect to such a thing the ordinary axioms of whole and part will remain consistent. Consequently DEDEKIND's system has no mathematical significance; in order to give it logical significance, an independent consistency proof would have been needed, but DEDEKIND does not give such a proof. If he had given it, he would necessarily have appealed to the intuition of 'and so on', and by recognizing this intuition, he would have seen that using it he could have constructed arithmetic in a very simple way, and from that moment on his logical system would have appeared to him gratuitous as well as cumbersome. He would not have maintained any longer that anybody who calculates, passes unconsciously through all the phases of his logical system. (See Vorrede, p. IX: 'Ich erblicke gerade in der Möglichkeit, solche Wahrheiten auf andere, einfachere, zurückzuführen, mag die Reihe der Schlüsse noch so lang und scheinbar künstlich sein, einen überzeugenden Beweis dafür, dass ihr Besitz oder der Glaube an sie niemals durch innere Anschauung gegeben, sondern immer nur durch eine mehr oder weniger vollständige Wiederholung der einzelnen Schlüsse erworben ist.')

On the same grounds as DEDEKIND's work, MANNOURY's paper: 'De zoogenaamde grondeigenschap der Rekenkunde' (Handelingen 8e natuur- en geneeskundig congres, Rotterdam 1901, p. 121–147), which elaborates more simply the same idea, must be rejected as a foundation of arithmetic. [See the reply to MANNOURY's criticism of BROUWER's thesis (1908B) [p. 105] and remark 8 in (1917) [p. 147].]

independence (i.e. in order to prove that some axiom cannot be logically deduced from some other axioms, a mathematical system is indicated such that the latter axioms can, but the former cannot be held to express properties of that system), and in this sense they consider, among other things, non-Archimedean and non-Pascalian geometries, such as we have constructed at the end of Chapter I. ¹⁾ Thus these rather inharmonic, painfully concocted systems achieve, because they represent a more restricted set of axioms, some sort of priority above the simple, lucid Euclidean geometry. ²⁾ Such disturbing consequences follow when language, which is a means, however imperfect, for the communication of mathematics, but which has nothing to do with mathematics as such, except as an accompaniment, is considered as essential for mathematics, and when the laws governing the succession of sentences, the logical laws, are seen as directives for acts of mathematical construction.

Of course the modern axiomaticians still keep to their purpose of seeing an application of their logical systems, and therefore they have only built up such linguistic systems as are suitable to accompany constructible mathematical systems. Now the following question arises: suppose we have proved by some method, without thinking of mathematical interpretations, that the logical system, built up out of certain linguistic axioms, is consistent, i.e. that two contradictory theorems can occur at no stage of development of the system; suppose further that afterwards we find a mathematical interpretation of the axioms (which of course will require the construction of a mathematical system whose elements satisfy certain given mathematical relations); does it follow from the consistency of the logical system that such a mathematical system *exists*? Such a conclusion has never been proved by axiomaticians, not even for the case where the given conditions involve that it is a mathematically constructible system that is required. Thus, for instance, it has nowhere been proved that a finite number, subjected to a provably consistent system of conditions, must always exist. ³⁾

¹⁾ The geometries indicated by HILBERT (Grundlagen der Geometrie § 34) and VAHLEN (Abstrakte Geometrie p. 42, 110) are special cases of the group of geometries which I constructed there.

²⁾ From our point of view we can see in HILBERT's pathological geometries no more than special generalizations of the Euclidean group of motions. These generalizations remain one-sided in this respect, that (in contrast to those of LIE) they are restricted to projective groups and that (also differently from LIE's work) they do not lead to the *general* group satisfying certain conditions; but the latter cannot succeed before, by an arbitrary restriction, a *definite* basic mathematical system is given, in which the group, that must satisfy certain additional intrinsic conditions, must be imbedded.

³⁾ A fortiori it is not certain that any mathematical problem can either be solved or proved to be unsolvable, though HILBERT, in 'Mathematische Probleme', believes that every mathematician is deeply convinced of it.

But for this question as well it is of course uncertain whether it will ever be possible to settle it, i.e. either to solve it or to prove that it is unsolvable (a logical question is nothing else than a mathematical problem).

138

139

140

141

142

[[4]]

[[5]]

But the assertion is certainly *not* true if the given conditions do *not* involve that the system must be constructible. Thus, for instance, according to HILBERT the properties which CANTOR formulated for the well-ordered set consisting of *all* the numbers of the second number class, are consistent, but the set does not exist in mathematics.

Ad 2°

Rejection of the logical foundations of set theory

143

In chapter I we have seen that there exist no other sets than finite and denumerably infinite sets and continua; this has been shown on the basis of the intuitively clear fact that in mathematics we can create only finite sequences, further by means of the clearly conceived 'and so on' the order type ω , but only consisting of *equal elements*¹⁾; (consequently we can, for instance, never imagine *arbitrary* infinite dual fractions as finished, nor as individualized, since the denumerably infinite sequence of digits cannot be considered as a denumerable sequence of *equal* objects), and finally the intuitive continuum (by means of which we have further constructed the ordinary continuum, i.e. the *measurable continuum*), but no other sets.

[[6]]

144

CANTOR and his disciples however think that they have knowledge of all sorts of further sets; their fundamental principle is (CANTOR, Grundlagen einer allgemeinen Mannigfaltigkeitslehre, Leipzig 1883 [Ges. Abh. p. 207]: 'Der Vorgang bei der correcten Bildung von Begriffen ist m.E. überall derselbe; man setzt ein eigenschaftsloses Ding, das zuerst nichts anderes ist, als ein Name oder ein Zeichen A und giebt demselben ordnungsmässig verschiedene, selbst unendlich viele verständliche Prädicate, deren Bedeutung an bereits vorhandenen Ideeën bekannt ist und die einander nicht widersprechen dürfen; dadurch werden die Beziehungen von A zu den bereits vorhandenen Begriffen und namentlich zu den verwandten bestimmt; ist man hiermit vollständig zu Ende, so sind alle Bedingungen zur Weckung des Begriffes A, welcher in uns geschlummert, vorhanden und er tritt fertig ins Dasein, versehen mit der intrasubjektiven Realität, welche überall von Begriffen nur verlangt werden kann; seine transiente Bedeutung zu constatiren ist alsdann Sache der Metaphysik.'

We see that it comes to about the same as the standpoint of the axiomaticians.

Above we have shown that this principle is unjustified, and therefore we now assert that the several paradoxes of the 'Mengenlehre', for which a solution is so fervently sought, have no right to exist; rather it would have been the duty of Cantorians, immediately to reject a notion which gives rise to contradictions, because it is certainly not built up mathematically.

Let us discuss some points in detail.

¹⁾ The expression 'and so on' means the indefinite repetition of *one and the same* object or operation, even if that object or that operation is defined in a rather complex way.

CANTOR's *second number class does not exist*.

We know that the definition of *well-ordered sets* after CANTOR (see Chapter I, p. 63 [p. 45]) is consistent; for there exist well-ordered sets, in the first place the ordertype ω of the sequence of finite ordertypes: 0, 1, 2, ... And further there is no objection to the introduction of ω as a new ordinal number and again starting to count

$$\omega, \omega + 1, \omega + 2, \dots, 2\omega, 2\omega + 1, \dots, m\omega + n, \dots$$

[7]

No more is there any objection to the introduction of ω^2 after all the numbers that can thus be formed; in this way we open a still larger domain of ordinal numbers, building a well-ordered sequence whose terms can be expressed by the general form

145

$$m_1 \omega^{p_1} + m_2 \omega^{p_2} + \dots \quad (p_r > p_{r+1}).$$

As the next ordinal number we can introduce ω^ω . In this way we can go on, and CANTOR shows (Grundlagen, p. 35) [Ges. Abh. p. 197] that every ordertype introduced by this method, and consequently the set of the numbers that have been introduced at each stage, remains denumerable. But then he continues:

‘Wir definieren daher die zweite Zahlenklasse als den Inbegriff aller mit Hülfe der beiden Erzeugungsprinzipie (he means by those two principles: *add one unit* and: *take for an ordertype ω the next higher element, the limit-element*) bildbaren, in bestimmter Succession fortschreitenden Zahlen:

$$\omega, \omega + 1, v_0 \omega^\mu + v_1 \omega^{\mu-1} + \dots + v_{\mu-1} \omega + v_\mu, \dots, \omega^\omega, \dots, \alpha, \dots$$

welche der Bedingung unterworfen sind, dass alle der Zahl α vorausgehenden Zahlen, von 1 an, eine Menge von der Mächtigkeit der ersten Zahlenklasse bilden.’

Take notice of ‘den Inbegriff aller’; here he mentions something which cannot be thought of, i.e. which cannot be mathematically constructed; for a totality constructed by means of ‘and so on’ can only be thought of, if ‘and so on’ refers to an ordertype ω of equal objects, but *this* ‘and so on’ does not refer to an ordertype ω , nor to equal objects. Here CANTOR *loses contact with the firm ground of mathematics*. According to his fundamental principle, which we cited above, he must be indifferent to this, but in any case he ought to take care that the introduction of ‘Inbegriff aller’ does not give rise to contradictions. He fails to do this, but it can be done by the method by which HILBERT introduces the logical entity ‘Inbegriff aller’ and proves its consistency. ¹⁾

146

Now CANTOR goes on and talks about his second number class as if he had it before his eyes as a real object; the way he expresses himself does not suggest at all that he has in mind no more than a logical system. In his proof of the theorem that the second number class has a higher power than the first one, more exactly

¹⁾ Verhandlungen internat. Mathematiker-Congress Heidelberg 1904, p. 183, 184. [Grundlagen der Mathematik, Anhang VII.]

the next higher power, he considers this equality, respectively inequality, of power as the possibility, respectively impossibility, of a one-to-one mapping between two *existing* number classes.

147 From our point of view these reasonings with the meaning which CANTOR puts into them, *make no sense*; the best we can make of it, changing it a little, is the following triviality: If the logical entity T (power of the second number class) is introduced, then the axiom $T = A$ (A is the power of ω) will lead to a contradiction in the logical structure; likewise the introduction of a logical entity I , playing the part of a cardinal number, which would be supposed to satisfy the axioms $A < I < T$. This is the logical result, without any value for mathematics, of these proofs by CANTOR. If one wishes to look at it in the light of mathematics, then one finds no more than the following: The next two statements are *false*:

1°. The second number class is conceivable and denumerable.

2°. The second number class is conceivable and there is a cardinal number between its power and that of the first number class.

But that these two statements are false, we knew already, for we knew that the first part of both (the second number class is conceivable) is false.

148 And as to the mathematical argument running parallel to the developments contained in CANTOR's proofs, there remains only the following: 'Certainly there is no sequence, having power A , of well-ordered sets such that I could not construct a new well-ordered set not belonging to that sequence. But the totality of well-ordered sets which I have introduced in some mathematical system, is certainly denumerable.' (In this *mathematical* theorem we do not use the expression 'numbers of the second number class', because here the word *class* does not awaken a clear notion; nor do we speak explicitly of the *denumerable* well-ordered sets, for we can construct no other well-ordered sets than denumerable ones.)

[[8]]

The denumerably unfinished sets

If we wish nevertheless to speak of 'the totality of well-ordered numbers' and to say something about its power, then this succeeds, in connection with the above theorem, if we construe it in a somewhat different sense, by the following statement:

[[9]] *The power of the totality of well-ordered numbers is denumerably unfinished*; here we call a set *denumerably unfinished* if it has the following properties: we can never construct in a well-defined way more than a denumerable subset of it, but when we have constructed such a subset, we can immediately deduce from it, following some previously defined mathematical process, new elements which are counted to the original set. But from a strictly mathematical point of view this set does not exist as a whole, nor does its power exist; however we can introduce these words here as an expression for a known intention.

149 As further examples of denumerably unfinished sets we mention: the totality of definable points on the continuum, and a fortiori the totality of all possible mathematical systems.

While constructing, without ever coming to an end, a denumerably unfinished set, we can map the elements in succession on the sequence of the well-ordered sets, which likewise is never exhausted. Extending the notion of equality in power, so that it may be applied to this case, we can say:

All denumerably unfinished sets have the same power. ¹⁾)

Thus we distinguish for sets the following cardinal numbers, in order of magnitude:

- 1°. the different finite numbers.
- 2°. the denumerably infinite.
- 3°. the denumerably unfinished.
- 4°. the continuous.

The continuum problem

The continuum problem, concerning which papers appear constantly with the aim of furthering the solution, requires a proof that the continuum and the 'totality of numbers in the second number class' have the same power. Now neither the totality of numbers of the second number class, nor the continuum as a system of individualized points, exist mathematically; therefore it appears from what we said above, that the only clear idea which we can find behind this problem, is the following logical theorem, which belongs outside mathematics proper:

'It is possible to introduce as logical entities the totality of numbers of the second number class and the totality of points of the continuum in such a way that it is non-contradictory to suppose a one-to-one mapping between them, leaving out no element of one of them.'

But if we introduce the logical entity: *totality* of the points of the continuum, abandoning the intuition of a continuum, we shall be forced to define *the points of the continuum*, and this is only possible by *definable laws of progression for approximating dual fractions*. Now in this sense the continuum is denumerably unfinished, and so is the second number class. Consequently the logical theorem is proved.

The related *mathematical* question (that all the sets which can be defined on the continuum are either denumerable or have the power of the continuum) has been treated in chapter I (p. 62–67) [p. 44–47].

The paradox of BURALI-FORTI

Following CANTOR further, we see how he introduces Ω as the first ordinal number

¹⁾ Still, in a certain sense, one can say that denumerably unfinished sets have the same power as denumerable sets, for every denumerably unfinished set can be mapped on ω^2 (every part which I add in the course of the construction of the denumerably unfinished set, can be mapped on ω , for it is denumerable; constructing such a mapping for each added part, I map the unfinished set on $\omega + \omega + \omega + \dots = \omega^2$; only this mapping remains *always unfinished*; the proof that a mapping of a denumerably unfinished set on a denumerable set is impossible, holds only for a *finished* mapping.

after all the ordinal numbers of the second class; he calls it the first ordinal number of the third number class. But *in mathematics* Ω does not exist; probably it is easy to give the logical proof for the consistency of the newly introduced object, but this is unimportant.

CANTOR's disciples undauntedly went still further on. They created as many number classes and cardinal numbers as they could create ordinal numbers, bothering neither about the question whether they were conceivable in mathematics, nor about their logical consistency. Finally they introduced *the totality of all ordinal numbers*, but now they noticed a logical contradiction, which, for that matter, they signalled as a mathematical paradox and for which they assiduously sought a *mathematical* solution (by mathematical we always mean: lying within the realm of what is intuitively conceivable), without being aware of the fact that they had long ago left the realm of mathematics.

152 It is BURALI-FORTI's paradox (Una questione sui numeri transfiniti, Rendiconti del circolo matematico di Palermo 11 (1897), p. 154-164): '*Let us denote by O the totality of well-ordered types, ordered by magnitude, then O itself is a well-ordered type, and because every well-ordered type occurs as a subset of O , O must be the largest well-ordered type. But if O is a well-ordered type, $O+1$ is so as well, and $O+1 > O$, consequently O is not the largest well-ordered type.*'

Firstly the paradox would be easily redressed if we did not assign to O the property (which was assigned only by arbitrarily assumed axioms to the well-ordered types which were created earlier) that $O+1$ is again a well-ordered type.

Secondly there is no reason to consider such a result as paradoxical: if we create logical structures without mathematical structures which they accompany as their linguistic counterpart, every such structure may be contradictory as well as consistent.

The well-ordering of an arbitrary set

Another famous problem from the theory of transfinite numbers is: '*Prove that any set can be well-ordered.*' CANTOR enounced this problem ('Grundlagen', p. 6 [Ges. Abh. p. 169]) as a 'Denkgesetz'; of course there is no motive for this at all, so his disciples tried to prove the theorem. In Math. Annalen 59 (1904), p. 514-516, ZERMELO gives such a proof based on the following axiom:

"Jeder Teilmenge M' einer Menge M kann man ein beliebiges Element m'_1 zugeordnet denken, das in M' selbst vorkommt und das 'ausgezeichnete' Element von M' genannt werden möge."

153 BOREL justly remarks in Math. Annalen 60 (1905), p. 194-195, that if one introduces such an axiom, he might as well take the theorem itself as an axiom.

Now we know that besides the denumerable sets, for which the theorem *certainly holds*, there exists only the continuum, for which the theorem *certainly does not hold*, firstly because the greater part of the elements of the continuum must be considered as unknown, and consequently can never be individually

ordered, secondly because every well-ordered set is denumerable. Thus this question also turns out to be illusory.

BERNSTEIN's theorem

The theorem of **BERNSTEIN-SCHRÖDER** is usually mentioned as the main theorem in the theory of transfinite numbers.

'Let A and B be sets and let A be mapped one-to-one onto a subset of B, and likewise B onto a subset of A, then A can be mapped one-to-one onto B',
 or, what comes to the same (we introduce the symbol ' $a \sim b$ ', to read: *a equivalent to b*, expressing that *a* and *b* can be mapped one-to-one onto one another):

If

$$\begin{aligned} A &= A_1 + B + C \\ A &\sim A_1 \end{aligned}$$

then also

$$A \sim A_1 + B.$$

(For let the theorem be proved in the latter form and suppose

$$\begin{aligned} A &= H_1 + C & H &= A_1 + D \\ H &\sim H_1 & A &\sim A_1, \end{aligned}$$

then we have also

$$H_1 = A_{11} + D_1$$

where

$$\begin{aligned} A_{11} &\sim A_1 \sim A \\ D_1 &\sim D. \end{aligned}$$

Now from

$$A = A_{11} + D_1 + C$$

there follows, by the second form of the theorem,

$$A \sim A_{11} + D_1 \sim H.)$$

This second form of the theorem is proved as follows: Let us apply the operation which divides *A* into a part, equivalent to the whole, and two other parts, another time to *A*₁, obtaining

$$A_1 = A_2 + B_1 + C_1,$$

then to *A*₂, etc., then we obtain finally:

$$A = B + B_1 + B_2 + \dots + C + C_1 + C_2 + \dots + D,$$

where *D* is the set common to all the consecutive *A*'s. Obviously

$$C + C_1 + C_2 + \dots \sim C_1 + C_2 + \dots$$

Consequently

$$A \sim B + B_1 + B_2 + \dots + C_1 + C_2 + \dots + D \sim A_1 + B;$$

and the required mapping is obtained by mapping C to C_1 , C_1 to C_2 , C_2 to C_3 , etc..

155 For sets which exist only as consistent logical entities this theorem shows that if

$$A = A_1 + B + C,$$

and a one-to-one mapping of A onto A_1 is given, then it is consistent to assume that also A and $A_1 + B$ are equivalent. Mathematically it also provides us with a method actually to construct a one-to-one mapping of A onto $A_1 + B$, but only for those elements of A which *have been defined*, which are *known*, i.e. for a denumerably unfinished part. For the unknown elements it does not furnish such a mapping. For instance, if A is a continuum, we shall for an arbitrary element, which is only known by an ever unfinished approximation, never know whether or not it belongs to one of the C 's, and if it does, to which C it belongs; consequently we can say nothing on the approximation of its image.

From the mathematical point of view the theorem, in the form in which we proved it above, makes sense only for finite, denumerable and denumerably unfinished sets. But for these it is evident.

[[12]] As we knew for a long time (see Chapter I, p. 62–67 [[p. 44–47]]), the theorem holds also for continua, but the proof given above is worthless in this case.

156 As the theorem turns out to be insignificant, we may expect that the many applications which the Cantorians make of it are equally devoid of content. Let us examine by way of example a paper by BERNSTEIN in Math. Annalen 61 (1905), p. 117–155. In order to bring the continuum problem nearer to its solution he derives next to the well-known theorem:

The cardinal number of all well-ordered types with cardinal number A (the first cardinal number) is F (the second cardinal number) an analogous theorem:

The cardinal number of all order types with cardinal number A is C (cardinal number of the continuum).

By means of his theorem on equivalence he deduces this theorem from the following two lemmas:

a) *The continuum is equivalent to a subset of the totality O_A of all ordertypes with cardinal number A .*

b) *The set O_A is equivalent to a subset of the continuum.*

He proves the former by assigning to an infinite dual fraction the ordertype obtained by inserting between each two consecutive digits an ordertype $\omega^* + \omega$, and then deleting every digit 0 and replacing every digit 1 by a single element.

157 His proof for the second lemma is incorrect. The method by which he constructs the ordertypes of cardinal number A (namely by placing first one element, then the second, for which there are two possible places, then the third, for which there are 3, etc.) never yields more than a special group of types, which is denumerable, namely $1 \times 2 \times 3 \times 4 \times \dots$. In the case of the dual fractions in the continuum a number A of operations could be thought of as finished, but this was

only possible on account of the *intuition* of the continuum; *here* an analogous intuitive possibility does not exist.

Consequently we can consider here only the ordertypes for which a *law* of progression is given, but then the set of *all* ordertypes is considered as a denumerably unfinished set of laws of progression. Thus the given mapping is from O_A considered as a set of laws of progression onto a part of the continuum.

Let us go back to the first lemma. We can read it in two ways, namely either: 'All points of the continuum are equivalent to a part of *all* members of O_A .' or:

'All laws of approximation for points of the continuum are equivalent to a part of *all* laws of approximation for members of O_A .'

It is only the second version that can be combined with the second lemma, insofar as it may be considered as proved by BERNSTEIN. But what does there now remain of the result? This, that *all* laws of progression in O_A are equivalent to *all* laws of approximation in the continuum, but this is obvious, for both sets are denumerably unfinished.

The transfinite exponentiation

Besides the sequence of well-ordered classes another method is employed in the theory of transfinite sets to ascend to higher cardinal numbers. It is based on exponentiation by a transfinite exponent.

By M^N there is meant the *insertion-set* of N by M , i.e. the set consisting of all manners in which we can make an element of M correspond to each element of N .

Then it can be proved that for $M > 1$,

$$M^N > N$$

(see for instance SCHOENFLIES, Bericht über die Mengenlehre, Jahresber. Deutsch. Math. Ver. 8 (1899), Heft 2, p. 26; there it is proved that $N^N > N$, but the proof can easily be extended to the theorem given above).

The first higher cardinal number obtained by this method is

$$C = M^A$$

where M is finite or denumerably infinite and A is the denumerably infinite power; we can conceive this insertion-set because we can conceive the continuum.

But we are already not able to conceive the next insertion-set of the sequence:

$$F = M^C,$$

thus the theorem that F (for instance the set of all functions of one real variable) $> C$, retains no more mathematical content than the following statement:

'It is possible to make a different function correspond to each element of C , but it is false that F is conceivable and that it can be mapped on C .'

But this we knew of course without the proof or the theorem, because we knew that F is inconceivable.

Propositional functions and classes

Classical logic was inadequate to account for mathematics. Logicians aimed at extending it in such a way that it would become equal to this task. Now we have seen that classical logic studies the linguistic counterpart ¹⁾ of logical reasonings, i.e. of reasonings on *relations of whole and part* for arbitrary mathematically constructed systems; from the fact that we *see* these mathematical systems we may conclude that *here* the sentences succeeding one another according to classical logic, will never show contradictions, because they correspond to acts of mathematical construction. Thus here we safely introduce the *logical sum*, the *logical product* and the *complementset*, i.e. the set having as its predicate the negation of the predicate of the given set; and we safely apply the *principles of identity, syllogism, distribution* ²⁾, *contradiction* and *tertium non datur*.

The logicians, conversely, start from these principles and as the fundamental domain of operations, within which the relations meant by the words or symbols must exist, they choose not some mathematical system, but the chimerical 'everything' – as we saw above (p. 138 seq. [p. 78], footnote), Dedekind also wrongly tried to start from this notion – from which they define various *classes* by what they call *propositional functions*.

By a propositional function they mean a *statement* about x , or about x and y , more generally about a certain number of variables, in which *every* substitution for these variables is allowed; they reckon that by that statement a *class* is defined, consisting of *all* things (or, for more variables, groups of things) which by substitution make the statement *true*.

They write $x\exists\phi x$ for *all things for which the statement ϕx is true*; the reciprocal sign ϵ is introduced in such a way that $k\epsilon(x\exists\phi x) = \phi k$, in other words: in the case that the propositional function determines a class a , $k \epsilon a$ means: *k belongs to the class a*.

PEANO preferred to call the sign ϵ primary; RUSSELL rather starts with $\exists$, because he considers it uncertain that every propositional function defines a class.

Herein he is right, however he works with his propositional functions as with the predicates of ordinary logic, so he acts all the same as if the function does always define a class.

But in the intellect one cannot give a linguistic system of statements and propositional functions priority over mathematics, for no assertions about the external world can be intelligently made besides those that presuppose a mathematical system that has been projected on the external world. No matter which

¹⁾ As well as any mathematical language this language can without much trouble be condensed into symbols. See for such a symbolical language (called 'Algebra der Logik') for instance A. N. Whitehead, A treatise on Universal Algebra, Cambridge University Press 1898, p. 35 seq.

²⁾ i.e. $(a+b)c = ac+bc$, where the logical sums and products are meant.

way one turns, the foundation of mathematics remains mathematics itself, and over its entire domain it grows freely and intuitively.

On the contrary the logicians, considering the propositional functions as the free origin of logic and mathematics, utter as such various sentences which are built in (falsely presumed) analogy to mathematical properties, and they postulate that these sentences define classes and that it is allowed to reason about these classes according to the laws of classical logic.

The contradiction

Therefore it is not surprising that they, just like the Cantorians, *came up against contradictions* ¹⁾ and their own surprise can only be due to confusion of ideas.

RUSSELL (The principles of mathematics, Part I, Chapter 10) discusses most amply the following contradiction:

‘There are classes which, considered themselves as one, occur amongst their own elements, for instance the class of all classes, the class of all things that are not alive, and others. Now I consider the class of all classes which do not possess the property which I mentioned just now, namely to occur amongst their own elements. Does this class possess the property in question? *If it does*, then it occurs amongst its own elements, so it is one of the classes which do not possess the property, so it does not have the property. And vice-versa: *If it does not*, then it is on equal terms with its own elements, so it does have the property.’

RUSSELL suggests various methods to escape from the contradiction, but he ends by rejecting them; he believes that a deep-searching reconstruction of logic will be needed for the solution. He is inclined to the opinion that a theory is required which does not admit every class, considered as one, to be made into a logical subject. ‘Another suggestion’, he says, ‘would be to demur to the notion of *all objects*, but in any case the notion of *every object* must be retained, for there are truths, viz. the logical principles, which hold for every object.’

But this is mistaken: the logical principles hold exclusively for words with a mathematical content. And exactly because RUSSELL’s logic is no more than a linguistic system, deprived of a presupposed mathematical system to which it would be related, there is no reason why no contradictions would appear.

For that matter, it is evident to common sense at which point the reasoning, which leads to the contradiction, ceases to be alive and consequently is no longer reliable; it is even unnecessary to give up the illusion of the chimerical ‘everything’. For let us suppose that I know an ‘everything’ with a ‘totality’ of relations existing between the objects, and a system of propositions which may hold for the objects. Then, given a propositional function, I can decide for *any object* by means of its given relations whether or not it satisfies the function, in other words, to which of the two classes defined by the function it belongs.

¹⁾ Obviously in such a contradictory system almost no reasoning is possible, because the main tool of reasoning, the principium contradictionis, can no longer be applied.

But when I wish to decide whether the object which is the class involved in the contradiction, satisfies the given propositional function, then I see that the decision is only possible under the condition that it has already been completed. Consequently the decision *cannot be taken*, and hereby the contradiction is explained. We have here a propositional function which defines two complementary classes which do not satisfy the tertium non datur. This is not surprising, for the logical principles hold only for the language of mathematics; for other linguistic systems, however akin to that of mathematics they may be, the principles need not hold.

RUSSELL gives the contradiction in another form on p. 80 and p. 102. There he says:

[[16]]

'There are predicates which hold for their own expression in words, and others which do not. The simplest example of the first kind is: *being a predicate*. But does this property hold for: *not holding for its own expression in words*? If it does, then it does not; if it does not, then it does.'

He tries to solve it by saying that *not holding for its own expression in words* is not a predicate, but of course nobody will assent to this.

And no more will anybody assent to what he proposes on p. 88, in order to escape from the contradiction in a third form, originating from the division of the propositional functions into such as do hold and such as do not hold for themselves, namely that it would be inconceivable to apply a propositional function to itself.

165

For that matter, the two latter contradictions can be explained in the same way as the first one.

The logic of relations

So far about the part played by *classical logic*; in logistics it is extended by the so-called *logic of relations*, and the final conclusion runs as follows: *Pure mathematics* can be nothing else than a system built up from a few *basic notions* of logic, by means of some *basic principles* of logic (RUSSELL counts 9 of the former and 20 of the latter); it is only in this way that it remains on solid ground and that it can be certain of progression. Perhaps there still exists some sort of intuitive mathematics, but then this consists in nothing more than the application of the pure mathematics to material objects (see e.g. COUTURAT, *Les Principes des Mathématiques*, Introduction, p. 4).

But we know that pure mathematics is neither the former, nor the latter.

The logic of relations, then, starting from the word *following after*, which expresses the most elementary act of mathematical construction, as it immediately springs from the basic intuition, studies the language of mathematics in general, just as classical logic studies that of the special mathematics of whole and part.

It is self-evident that in the language which accompanies mathematics, the succession of words obeys certain laws, but to consider these laws as directing the building up of mathematics, it is therein that the mistake lies.

Let us, by way of illustration, go into the theory of the positive integers, i.e. ordinary arithmetic, as it is treated by the logisticians.

PEANO (Sul concetto di numero, *Rivista di Matematica* 1 (1891), p. 87–102, 256–267; see also COUTURAT, *Les Principes des Mathématiques*, p. 54) introduces in addition to classical logic *three* new basic notions: 0, *N* (finite ordinal number) and *seq* (successor), and *five* basic principles which hold for them:

1°. 0 is an *N*.

2°. there is no *N*, for which 0 is the *seq*.

3°. the *seq* of each *N* is an *N*.¹⁾

4°. Two *N*'s are equal if their *seq*'s are equal.

5°. A class which contains 0 and which, with every *N* which it contains, contains also its *seq*, contains every *N*.

But in his deduction of arithmetic from these principles, PEANO again edifies a logical system which is supported neither by an existence proof, nor by a consistency proof.

Therefore it must be rejected on the same grounds as DEDEKIND's system (see p. 138 *seq.* [p. 78], footnote).

Arithmetic of RUSSELL

RUSSELL (The Principles of Mathematics, p. 127) considerably improves PEANO's method. He starts by defining *cardinal numbers* as classes of equivalent classes, and he says further:

1°. 0 is the class of classes whose only member is the null-class; the null-class itself is defined (i.e. p. 75) as the class of all class-concepts which give no members for the corresponding class and therefore it is considered (or postulated as a principle?) to exist – consequently the null-class exists.

2°. 1 is the class of all classes which are not null and are such that, if *x* belongs to the class, the class without *x* is the null-class.

The class 1 exists, for it has already a member belonging to it, namely the class 0 defined just now.

3°. *n* + 1 is the class of classes which are equivalent to a class, obtained by adding a number to a class belonging to the class *n*. Consequently the class *n* + 1 exists, provided *n* exists.

4°. Finite numbers are those belonging to every class *s* to which belongs 0 and to which *n* + 1 belongs if *n* belongs.

All PEANO's postulates are satisfied by the class of finite numbers as above defined; it was unnecessary to introduce new basic notions or basic principles. Therefore, according to RUSSELL, they can serve as an existence proof (if we consider *n* + 1 as the *seq* of *n*). COUTURAT (reply to POINCARÉ, *Revue de Méta-*

¹⁾ Here it ought to be added (POINCARÉ, *Revue de Métaphysique et de Morale* 13 (1905), p. 833): every number has a successor, every *N* has a *seq*.

physique et de Morale 14 (1906), p. 208–250) strongly emphasizes this existence proof which, he contends, does not need the intuition of ω , or of complete induction, so that the logical system would here be built up independently of intuition and would have been shown consistent without the use of complete induction.

168

But we have found above that it is contradictory to assume that every definition determines a class, therefore the consistency of a logical system can never be based on the fact that it runs parallel to pseudo-mathematical operations in classes, which only exist by their definitions.

All the same RUSSELL dauntlessly applies complete induction, which he refuses to enounce as an axiom. Namely he proves that $n+1$ is the cardinal number of the class of the numbers 0, 1, 2, ..., n , as follows: 1 is the cardinal number of the class 0; 2 (defined as $1+1$) that of the class 0, 1; 3 (defined as $2+1$) that of the class 0, 1, 2; and so on.

Likewise he applies complete induction to prove that a finite number cannot be equivalent to one of its parts (l.c. p. 121, 123).

[[18]]

Infinite numbers

After the finite numbers have been defined, they yield immediately the definition of the cardinal number A of the class which contains all of them (the first transfinite cardinal number). RUSSELL proves easily that this cardinal number is not finite, because it does contain parts equivalent to the whole; he further shows that any infinite class contains parts with cardinal number A (l.c. p. 122, 123).

But the way in which the logisticians further develop mathematics, no longer displays special features, except that the language *is condensed* as far as possible *in symbolic signs*. It merges in the methods of Cantorians and of axiomaticians.

169

Conclusions on logistics

The conclusions on logistics must be: It can teach us nothing about the foundations of mathematics, because it remains irrevocably separated from mathematics; on the contrary, in order to maintain an existence on its own account, i.e. to safeguard itself against contradictions, it must reject all its own special principles and acquiesce to be a faithful, automatic, stenographic copy of the *language of mathematics*, which itself *is not mathematics*, but no more than a defective expedient for men to communicate mathematics to each other and to aid their memory for mathematics.

Ad 4°

Consistency proofs for formal systems, independent of their interpretation

The most uncompromising conclusion of the methods we attack, which illustrates most lucidly their inadequacy, has been drawn by HILBERT (Verhandlungen Intern. Math. Congress Heidelberg 1904, p. 174 [Grundlagen der Geometrie, An-

hang VII]). In 'Ueber den Zahlbegriff' (Jahresber. Deutsch. Math. Ver. 8 (1899), p. 180–184 [Grundlagen der Geometrie, Anhang VI]) he had formulated the axioms for the arithmetical operations on the measurable continuum and put forward the problem of proving the consistency of these axioms independently of any mathematical intuition (see also 'Mathematische Probleme', Problem no. 2, Göttinger Nachr. 1900, p. 264 [Ges. Abh. III, p. 290–329]).

170

Evidently this can only be attained by considering the very signs which express the axioms as a mathematical system, by formulating the principles of logic, in the manner of the algebra of logic, as rules allowing to extend this system, and by proving mathematically that these rules taken from the algebra of logic can never lead to an equation together with its negation. Obviously the set of equations deducible from the basic equations formulated in the axioms, is denumerably infinite.¹⁾

HILBERT loc. cit. sketches in outline how to realize these consistency proofs²⁾, not only for the axioms describing the arithmetical operations, which I mentioned just now, but also for those of various other parts of mathematics. For instance, in order to lay the foundations of set theory, he introduces on p. 182–184 the class symbol, but only in relation to other symbols which were introduced before; thus he protects himself from the contradictions obtained by RUSSELL, who introduced classes as parts of the universe, delimited by definitions.

171

HILBERT however, as he says explicitly in his introduction, aspires to start from nothing and to develop mathematics and logic together. But in the reasonings, mentioned above, on the consistency of axioms, he applies again and again intuitive terms such as *one, two, three, some* (by which he means *a certain finite number*), and further he intuitively applies all the laws of logic and even complete induction.

Attempt to make these proofs independent of intuition

In order to get rid of this burden of intuitive notions, he ends (loc. cit. p. 184, V) by suddenly looking back at the words which he had written, considering this complex of words and reasonings as a mathematical structure which again develops from the beginning to the end according to certain rules, and he contends:

'I have proved just now that the rules from which I have seen this linguistic structure develop, are consistent and therefore correct. In other words, the reasonings which I have made in that language justify at the same time the intuitive element in the act by which they were made.'

[[19]]

¹⁾ For the set of all combinations of a finite number of the signs which have been introduced (in which the sign = is included and which are finite in number for each mathematical theory) remains denumerable, a fortiori this holds for the set of those special combinations of signs which may be read as *true* equations.

²⁾ In one place he makes a mistake, namely on p. 181, where he gives a consistency proof by means of an *example*, which is of course inadmissible from his point of view.

This is mistaken for the following reasons:

172

Firstly the ground on which he stands *remains* the intuition under discussion, for he only knows: *provided* this intuition is correct, then it follows that the words accompanying it develop according to a consistent logical system. But this is nothing new; who will try to prove a mathematical theorem by deducing it once more from the theorem itself, and say afterwards: 'Now the hypothesis of the theorem is justified at the same time'?

But conversely: The consistency of the linguistic system, deduced by means of the mathematical intuition, does not prove the mathematical intuition. This we have shown above when we treated the axiomatic foundations (see also POINCARÉ, *Revue de Métaphysique et de Morale* 13 (1905), p. 834).

173

The method which we reject here surpasses that of the logicians in that it takes, several times in succession, the unjustified step from the original mathematical domain to another by means of the linguistic act, and because it does not, as the logicians do, maintain the intuitive connection between the two mathematical domains which are connected only through the intermediary of language. It continues to treat them as dissimilar, but in the end it confuses them and puts them on the same level. The logicians take the step once and then operate alternately on both domains, maintaining the significance of each; HILBERT takes the step decisively and definitively and *remains* on the second level, returning to the first only to give it a sense in the second, then he takes a further step, again definitively, *remains* on the third level which he so creates, and uses the first and second only to give them a sense in the third.

[[20]]

We explain this more amply by enumerating in genetic order the different stages which can be distinguished:

Enumeration of the stages which are confused in the logical treatment of mathematics

1. The pure construction of intuitive mathematical systems which, if they are applied, are turned outward in life by taking a mathematical view of the world.

2. The linguistic parallel to mathematics: mathematical speaking or writing.

3. The mathematical study of language: we notice logical linguistic structures, raised according to principles from ordinary logic or from its extension by the logic of relations, i.e. logistics, but the elements of these linguistic structures accompany mathematical structures or relations.

4. Forgetting the sense of the elements of the logical figures mentioned just now, and imitating the construction of these figures by a new mathematical system of *second* order, provisionally without a language parallel to the construction; this is the system of the logicians, which, if it is in the least generalized by a free extension, is very well pervious to contradiction, unless HILBERT's precautions are taken; these precautions form the main content of HILBERT's paper.

174

5. The language of logistics, i.e. the words which accompany the logical con-

struction and account for it. It is true that PEANO tried as far as possible also to connect the thoughts which accompany the construction with symbolic signs; nevertheless it remains possible to split up his system into the pure structure and the principles by which the structure develops. Even when these principles themselves are also formulated by means of symbols, such formulations must be considered as heterogeneous with respect to the other formulas. The former are applied to the latter not in their capacity of formulations, but in that of intuitive acts, for which the formulations are no more than a linguistic counterpart.

HILBERT needs these intuitive acts, and therefore the language connected with them, more than PEANO, because he aims at proving the consistency of the logical system *in itself*, a problem which PEANO does not care about.

The verbal content of HILBERT's paper up to p. 184, V, belongs to the fifth stage.

6. The mathematical study of this language. This step is essential in HILBERT's work, in contrast to that of PEANO and RUSSELL. He observes, looking back at his own words, logical figures developing after logical and arithmetical principles, among them the theorem of complete induction; the elements of these logical figures, such as the words *mehrere*, *zwei*, *Fortsetzung*, *an Stelle von*, *beliebig*, etc. correspond in the language to acts of construction in the mathematical system of the second order mentioned above.

175

7. Forgetting the sense of the logical figures, and imitating their construction in a new mathematical system of the third order, provisionally without a corresponding language.

HILBERT performs the transition from 6 to 7 *in his mind* loc. cit. p. 184–185, under V, first paragraph.

8. The language corresponding to the mathematical system of the third order, which accounts for the construction of that system and shows that it is consistent.

This stage is, in the *words* of the paragraph on p. 184–185, mentioned just now, the last to be found in HILBERT's paper.

It would be possible to go on still further, but the mathematical systems of higher order would almost be copies of one of them, therefore it does not make sense further to continue the process.

For that matter, even the stages mentioned above, from the third on, are deprived of mathematical significance. Mathematics has its place only in the first; in practice it cannot remain aloof from the second, but this stage remains an unconscious non-mathematical act. It may afterwards be guided and supported by *applied mathematics*, but it can never obtain priority relative to intuitive mathematics.

POINCARÉ's criticism

Logistics and Cantorism have already been sharply criticized by POINCARÉ (*Revue de Métaphysique et de Morale* 13 (1905), p. 815–835; 14 (1906), p. 17–34,

176

294–317, 866–868). He blames mainly in logistics the *petitio principii* and in Cantorism the hypothesis of the actual infinite. But hereby he does not touch upon the heart of the question, which is situated deeper, namely in the confusion between the act of constructing mathematics and the language of mathematics.

In a sense the *petitio principii* is allowable, for where it occurs in the act of construction of the linguistic system, it does not as such affect the perfection of that linguistic system; in mathematics we would only have an unjustified *petitio principii* where on the basis of a primary mathematical intuition the same intuition would appear in a later phase of the mathematical construction, if there it were presented as not primary.

But the mistake of logistics lies in the fact that it creates nothing else than a linguistic structure, which can never be transformed into mathematics proper.¹⁾

And as to the actual infinite of the Cantorians, it does exist, provided we confine it to that which can be intuitively constructed, and refrain from extending it by logical combinations that cannot be realized.

177 How far POINCARÉ is from taking the intuitive *construction* as the only basis of his criticism, appears from his words (loc. cit. p. 819):

‘Les mathématiques sont indépendantes de l’existence des objets matériels; en mathématiques le mot exister ne peut avoir qu’un sens, il signifie *exempt de contradiction*.’

It might almost have been written by his opponent RUSSELL. It is true that mathematics is quite independent of the material world, but *to exist* in mathematics means: to be constructed by intuition; and the question whether a corresponding language is consistent, is not only unimportant in itself, it is also not a test for mathematical existence.

Making a mathematical study of linguistic symbols, no matter whether they are words or Peanean symbols, can teach us nothing about mathematics; mathematical formulas ought not to be considered as ‘truths’ existing independently, but only as an expedient to remind us as efficiently as possible by means of symbols of the way in which a certain structure was imbedded in another structure. For instance, the formula

$$13 = 6 + 7$$

reminds us of the fact that a set originating by juxtaposition of two sets along which we could count up to 6, respectively up to 7, was imbedded in a set along which we could count up to 13.

¹⁾ Obviously the *petitio principii* does become unjustified as soon as one tries, like HILBERT, to reason from the linguistic system to the primary intuition which it accompanies.

SUMMARIZING:

179

Mathematics is created by a free action independent of experience; it develops from a single aprioristic basic intuition, which may be called *invariance in change* as well as *unity in multitude*.¹⁾

Further, projecting mathematical systems on experience is also a free action, which shows itself efficient in the struggle for life; in this respect one mathematical system can appear more practical, more economical than another, at least relative to a definite kind of purpose which one wishes to attain: none of them is absolutely efficient, neither Euclidean geometry, nor logical reasonings, nor the theory of electrons.

180

In mathematics mathematical definitions and properties ought not to be studied again by mathematical methods; they ought to be no more than a means of conducting as economically as possible one's own memory and communication with other people. In the system of definitions there are elements of mathematical construction which must remain irreducible and which therefore, when communicated, must be understood from a single word or symbol; these are the elements of construction which are immediately conceived in the basic intuition or intuition of the continuum. Notions such as *continuous*, *entity*, *once more*, and *so on* are irreducible.

[[21]]

A logical construction of mathematics, independent of the mathematical intuition, is impossible – for by this method no more is obtained than a linguistic structure, which irrevocably remains separated from mathematics – and moreover it is a *contradictio in terminis* – because a logical system needs the basic intuition of mathematics as much as mathematics itself needs it.

[[22]]

¹⁾ The first act of construction has *two* discrete things thought together (also according to CANTOR, Vortrag auf der Naturforscherversammlung in Kassel 1903); F. Meyer (Verhandl. internat. Math. Congr. Heidelberg 1904, p. 678) says that *one* thing is sufficient, because the circumstance that I think of it can be added as a second thing; this is false, for exactly this *adding* (i.e. setting it while the former is retained) *presupposes the intuition of two-ity*; only afterwards this simplest mathematical system is projected on the first thing and the *ego which thinks the thing*.

[[97]]

STATEMENTS

[[to be defended together with the thesis]]

I

SCHUBERT (Enz. der Math. Wiss. I A 1 § 1) is wrong where he states: 'Dinge zählen heisst: sie als gleichartig ansehen, zusammen auffassen, und ihnen einzeln andere Dinge zuordnen, die man auch als gleichartig ansieht.' ... 'Wegen der Gleichartigkeit der Einheiten unter einander ist die Zahl unabhängig von der Reihenfolge, in welcher den *Einheiten* die *Einer* zugeordnet werden.'

II

It is not only impossible to prove the admissibility of complete induction, but it ought neither to be considered as a special axiom nor as a special intuitive truth. Complete induction is an act of mathematical construction, justified simply by the basic intuition of mathematics.

III

The language of Euclidean geometry, and likewise that of the corrections which PASCH and HILBERT have made therein, is reliable only because the mathematical systems and relations, which are symbolized by the words of that language as conventional signs, have been constructed beforehand independently of that language.

From this point of view Euclidean geometry, and likewise the different pathological geometries of HILBERT, consist in the study of groups of transformations with given properties, which can be applied to certain mathematical systems. [(1907), p. 75–80].

IV

KLEIN's defence (Zur ersten Verteilung des Lobatcheffsky-Preises, Math. Annalen 50) of the restriction to analytical transformations in LIE's research on the foundations of geometry lacks foundation.

[(1907), p. 30, 40; Math. Annalen 67 (1909), p. 246–267 [Vol. 2, 1909 C]].

V

The arithmetical operations on the measurable continuum ought to be defined by means of group theory.

[(1907), p. 19–30; Math. Annalen 67 (1909), p. 246–267 [Vol. 2, 1909 C]]].

VI

In physics a distinction between phenomenological and theoretical reasonings cannot be maintained. In particular there is no fundamental difference between the explanation of the properties of gases and liquids by molecules and that of the properties of light by electrical vibrations.

VII

Attributing 'objectivity' to physical notions like *mass* and *number* is based upon their invariability with respect to an important group of phenomena in the mathematical image of nature.

[(1907), p. 59–60].

VIII

Human understanding is based upon the construction of common mathematical systems, in such a way that for each individual an element of life is connected with the same element of such a system.

IX

Mathematics is independent of logic; practical logic and theoretical logic are applications of different parts of mathematics.

[(1907), Chapter 3].

X

Logical reasonings about the world can only be secure when they are connected with mathematical systems that have been previously constructed and projected on the world. The contradictions in logistics must be explained by the lack of such a system, KANT's antinomies by the fact that he applies different mathematical systems in the course of one and the same reasoning.

[(1907), Chapter 3].

XI

HOUËL (Cours de calcul infinitésimal, tome I, pag. 3) is wrong where he states:

'Une Science fondée sur des hypothèses qui sont compatibles entre elles, et qui ne sont pas réductibles à un moindre nombre, est absolument vraie au point de vue rationnel et abstrait, quand même elle ne se trouverait pas conforme aux faits réels qu'elle était destinée à représenter'.

XII

Besides the finite there are no other cardinalities than:

denumerably infinite

ever denumerable, ever unfinished

continuous.

[(1907), p. 83].

In the review by Mannoury (Nieuw Arch. Wisk. (2) 8 (1907), p. 175–180) there occur the following points which require a reply in order to prevent misunderstanding about the purport of the book under review.

[[1]]

1°. On page 176 Mannoury writes:

[[2]]

‘He who considers mathematics as produced by the human intellect, must be the first to acquire the conviction that the components of this mental complex are intimately connected with the nature of that intellect. Thus in most of the research on the foundations of mathematics done in the last half century the main question was not whether, but where and how mathematics takes its origin from (internal or external) experience. Therefore it seems to me that even if we admit the intuition of the continuum as the alpha and omega of all mathematics, this does in no respect affect the soundness of the mathematical-logical method. While therefore the author’s defense of his point of view seems least strong against the more rigorous symbolists (Dedekind, Peano), there is one remark made by the author which we are glad to acknowledge as perfectly justified. Namely, where he blames the symbolists for lavishly using, in the consistency proofs which they usually join to their systems of formulas, examples taken from (mathematical) experience, thereby suddenly deviating from the formal course. However, the author again goes too far where he concludes from this bad practice of many symbolists (as, for instance, Dedekind in “Was sind und was sollen die Zahlen”) that the consistency proofs cannot be supplied *without* such examples from experience, in a purely formal way. This is in my opinion one of the most important problems which symbolic logic has still to solve.’

The reply to this last remark must be that the considerations in ‘Over de grondsiagen der wiskunde’ [the thesis] pp. 138, 139, 170, 171 [p. 78, 93] most positively purport to showing explicitly, albeit briefly, that such purely formal consistency proofs are impossible.

2°.

[[3]]

3°. On page 179 Mannoury bases some considerations about the main content of the book on the following contradiction which he has found: ‘The author mentions as an example (of a denumerably unfinished set) the set of definable points on the continuum, i.e. the set of the points or numbers which can be defined individually by a finite number of symbols, no matter whether digits, signs or words (for instance $\sqrt{3}$, e, etc.). However, as the author justly remarks in another place, this set is denumerably infinite.’

But this contradiction is no more than apparent. For the denumerably infinite system mentioned on p. 170 (footnote) [p. 93] of 'Grondslagen' is that of the possible combinations of a *finite* number of symbols which have been previously introduced, while for the denumerably unfinished system of the definable points on the continuum at any moment new symbols can be introduced, which may replace, if it is useful, an infinite number of symbols which were introduced before.

4°. On page 180 Mannoury writes:

'So we suppose that the author will be obliged to recognize that his reasoning is insufficient, when he tries to apply it to a concrete example, for instance to prove that the set of all (continuous and discontinuous) functions $y = f(x)$ belongs not to the third, but to a lower number class.'

Indeed the author will not try to prove that the set of all functions $y = f(x)$ belongs to a lower number class than the third, for this reason that 'set of all functions $y = f(x)$ ' as well as 'third number class' are senseless combinations of words.

Finally it must be remarked that the intention of the work under discussion is not to propose considerations about which different persons can have different opinions, but to establish truths which, just like mathematical truths, anybody who has once understood will forever affirm.

Today the following remarks must be made in relation to my book ²⁾ 'Over de Grondslagen der Wiskunde' (Amsterdam, Maas & Van Suchtelen 1907), which appeared ten years ago [[1907]].

1. The group-theoretic characterisation of the arithmetic operations on the measurable continuum (p. 12–35) [p. 19–30] has been developed in detail in *Math. Ann.* 67 (1909), p. 246–267 [Vol. 2, 1909 C]. In particular one will find there on page 258 the proof of the properties mentioned in the book at the bottom of p. 20 and at the bottom of p. 22 [p. 23 and 24], regarding the fixed points of the component one-parameter groups ³⁾; on pages 260–262 the justification of the transition on p. 25 [p. 25] from mere differentiability of the increment functions to the application of Lie's fundamental formula; on pages 264–265 the solution of the problem concerning the projective group, proposed on page 35 [p. 30].

2. The deduction of the arc length for the two-dimensional Euclidean and non-Euclidean geometries following Riemann's conception, which was performed on pages 43–46 [p. 34–36] by differentiations ⁴⁾, has been accomplished by synthetic methods in *Handelingen Nederl. Nat. Geneesk. Congr.* 12 (1909), p. 192–196 [Vol. 2, 1909 E]. The same result has been reached earlier by means of other infinitesimal methods by W. Killing (*Einführung in die Grundlagen der Geometrie I*, Paderborn 1893, p. 80–89) and by C. Flye-S^{te} Marie (*Théorie des parallèles*, p. 12–19). The deduction of the n -dimensional from the two-dimensional arc length has been developed in more detail in *Handelingen Nederl. Nat. Geneesk. Congr.* 12 (1909), p. 196–199.

3. The consequent sustainment of the intuitionist point of view (see (1914), p. 79) [p. 140] demands that the third of the construction principles for sets of points, mentioned on pages 63–66 [p. 45–46] and also in (1908 A) must be deleted. In the second principle not only ramifications with two branches, but also with any finite or denumerable infinite number of branches, must be admitted for the infinite tree to which the operation 'completion to the continuum' is applied after the ramifi-

[[1]]

¹⁾ Reprinted from *Verslagen Kon. Akad. Wetensch.* Amsterdam 25 (1917), p. 1418–1423, observing the erratum to that publication.

²⁾ A discussion on this book has been held by G. Mannoury and the author. See *Nieuw Archief Wiskunde* (2) 8 (1908), p. 175–180, 326–328. [A translation of the most important parts of Brouwer's reply to Mannoury is included in this volume (1908 B)]

³⁾ Thus the deduction of the latter property from the former, given on pages 21–22 [p. 23–24] becomes superfluous for the proof.

⁴⁾ Where differential quotients are used in this deduction, their existence is obvious.

cation has been carried out ⁵⁾). Hereby the first construction principle becomes a special case of the second. The solution of the continuum problem which follows from the construction principles, remains valid after these changes. In each of the three places cited above, and also in (1912A), p. 92–93 [p. 134–135] the evidence for the denumerable or continuous cardinality of a set of points (and also for the possibility of splitting a closed set into a perfect and a denumerable one) are based upon two essential suppositions, namely in the first place that the set can be constructed in such a way that it is individualized, i.e. so that different infinitely proceeding branches of the tree produce different points, and further that the individualized pointset can be *internally dissected*, i.e. that the process of breaking off the branches which do not ramify any more, which must terminate after a denumerable number of steps, really can be effected. Now it is true that from the intuitionist point of view the unrestricted comprehension axiom cannot be used (see below under 7); therefore it is impossible to avoid special hypotheses about the way in which the pointsets under consideration are constructed, and thereby about the limitation of the domain of set theory. This implies the right to consider as contained in the construction principles such hypotheses as are desirable for the viability of the theory. However it has lately become clear to me, as I hope to explain in a paper that will shortly appear ⁶⁾, that the limits of set theory can be extended leaving out the two suppositions, mentioned above, on the construction principles, whilst preserving the viability of the theory. As a consequence the solution of the continuum problem turns out the other way. An example of a subset of the continuum with a cardinal number between the denumerable and the continuous is given in (1912A) p. 92 in a footnote [p. 134, footnote †].

4. In the construction of examples of non-Archimedean operations on page 68, line 4 [p. 47] in stead of: ‘if all their indices are greater than those of a given coordinate’ should be read: ‘if, given the preceding indices, every index is greater than a given number’. Further, on page 69, lines 1, 3 and 12 [p. 47,48] in stead of ‘index’, ‘indices’ should be read ‘number’, ‘numbers’, and on page 71 line 13 [p. 49] the letters q and r must be interchanged. Meanwhile extensive possibilities for the construction of commutative non-Archimedean operations have been developed by Hahn (Wiener Sitzungsber. 116 (1907), p. 601–655).

5. The deduction on pages 85–88 [p. 55–56] of the differentiability of a function from the continuity of the system of its difference quotients has been developed in more detail and extended to higher differential quotients and to functions of several variables in Proc. Nederl. Akad. Wet. 11 (1908), p. 59–66 [Vol. 2, 1908 D].

6. The point of view about objectivity and apriority of space and time defended on pages 95–99 and 118–121 [p. 59–61 and 69–71] has been confirmed by the

⁵⁾ As a matter of fact this was already done in a special manner on p. 158 [p. 87].

⁶⁾ (1918)

services which the theory of relativity has rendered to physics. See the considerations on this point in (1909), [p. 112–120], in particular as an illustration of the necessity of the definition of objectivity on page 96 [p. 60].

7. In the argument that logic depends upon mathematics, on pages 131–132 [p. 74–75] the applications in mathematics of the principle of *tertium exclusum* (or principle of *tertium non datur*) are called tautologies giving no information at all. This is an error; on the contrary these applications are often unjustified petitiones principii, as has been explained in (1908C), p. 152–158 [p. 109–111]. Further on page 135 [p. 76] the comprehension axiom (saying that the set of all mathematical entities having a certain property exists) is restricted to mathematical entities belonging to a previously constructed mathematical system *s*. Here we must add, in agreement with page 177 [p. 96], the condition that the set postulated in the axiom must itself be a constructible mathematical system. (The condition formulated by Zermelo in Math. Ann. 65 (1908), p. 263, which according to him is essential for modern formalism, namely that it must be determined for every entity in *s* whether the property in question holds for it or not, is from the intuitionist point of view much too weak or much too strong, depending on whether the *solubility of all mathematical problems* is accepted or not.)

8. Mannoury's paper cited on page 140 [p. 78] has in the mean time been reproduced with minor changes in his book 'Methodologisches und Philosophisches zur Elementarmathematik', p. 59–72 and a paper containing a very similar reasoning on the same subject has been published by Zermelo in Acta Mathematica 32 (1909), p. 185–193. Both argumentations suffer from the mathematical mistake that they use without restriction the comprehension axiom and the principium tertii exclusi, and from the logical mistake that they do not contain a consistency proof. Moreover Mannoury's argument postulates on p. 71 the *existence* of an O–R, namely that of the written ordinal numbers; this postulate is equivalent with admitting the intuition of complete induction. Finally he assumes at the top of p. 72 without proof that a set *a* containing a subset equivalent to the set of all subsets of a set *b* also contains a subset equivalent to *b*⁷⁾; while Zermelo l.c. p. 190 needs for this proof the logical petitio principii expressed in the *axiom of choice* (see (1912A1), p. 15 [p. 129]).

9. Concerning the discussion on the second number class on pages 144–149 [p. 81–82] see (1912A2), p. 91 [p. 133]; in the footnote the cited expressions: 'muss es geben' and 'können wir bestimmen' have been interchanged by a confusing printer's error.

10. Concerning the logically and mathematically precise formulation of the continuum problem given on pages 149–151 [p. 82–83] and the corresponding solutions, see (1912A2), p. 91 [p. 133]; note the restriction mentioned above under 3., regarding the solution of the first formulation given l.c..

⁷⁾ Another objection that was made in (1911) against the same passage can easily be obviated by a small modification to the proof.

11. Concerning the well-ordering problem it can be remarked (in agreement with (1914), p. 81 [p. 142], where however the wording is not so exact as might be desired) that a non-denumerable set of points, defined by an infinite tree, can only be linearly ordered in the sense that successively a fundamental sequence of finite sets $i_1, i_2, \dots$ of finite branches which are not continuations of each other, is linearly ordered; in this process it must be certain for each k that $\mathcal{S}\{i_k, i_{k+1}, \dots\}$ contains an initial segment of every infinite *final* branch. Let us denote by j_k the denumerable set of finite branches, obtained by deleting from $\mathcal{S}\{i_k, i_{k+1}, \dots\}$ every branch which is a continuation of a preceding branch in this set; it can be proved in each branch τ of j_v for at most one point (namely the point determined by choosing for $\mu > v$ in each consecutive j_μ the *last* continuation in the order) that it has a next element in the resulting order for the pointset. Consequently the set of the points for which this can be proved cannot have a higher cardinality than the denumerable; therefore a point set of higher cardinality than the denumerable can certainly not be well-ordered.

12. The criticism of Bernstein's equivalence theorem has been made precise and illustrated by an example in (1912A), p. 94–96 [p. 136–138]. The criticism of the mapping, defined by Bernstein, of the denumerable order types on points of the continuum (p. 156–157 [p. 86–87]) can be better formulated as follows. Bernstein's method does not provide us with a means of constructing the point corresponding to a given denumerable order type; consequently the mapping can only be applied to the order types for which a definite choice has been made among the possible ways in which the order relation can be gradually determined parallel to the enumeration of its elements; these order types form at most a denumerable set.

[[2]]

13. The proposition discussed on pages 158–159 [p. 87], that F (the set of functions of a real variable) has a higher cardinality than C (the set of points of the continuum) is sometimes interpreted as follows: 'It is possible to make different functions correspond to all the points of the continuum, but to any infinite tree of such functions a function can be found that does not belong to it' (see (1912A), p. 94 [p. 136]). But by closer inspection this interpretation also turns out to be untenable, namely because of the circumstance that a discontinuous function, defined in the Cartesian XY -plane, does not necessarily define a function of x on the line $x = y$.

[[3]]

14. On page 160 [p. 88] it is stated that even with respect to infinite mathematical systems the *logical sum*, the *logical product* and the *complement set* can always be safely introduced; this is based on the form of the comprehension axiom which has been criticized above under 7.; therefore it cannot be maintained. Likewise the principle of *tertium non datur*, which was maintained on the same page, and also on pages 163–164 [p. 89–90] for mathematical subjects and predicates, as opposed to non-mathematical subjects and predicates, has been rejected above under 7., even for mathematical systems.

15. With respect to Russell's arithmetic it must be remarked that though the reasonings discussed on page 168 [p. 92] ('The Principles of Mathematics' p. 121–123) are based on the intuition of complete induction, the author explicitly stresses that these reasonings are only provisional; indeed the properties in question can also be derived independently of this intuition on the basis of the system of definitions cited on page 167 [p. 91]. But this does not corroborate Russell's standpoint, for this system of definitions can only be applied after the addition of a postulate saying that at least one class satisfying the conditions of the last axiom exists; such a postulate is not implied by the comprehension axiom and it is practically equivalent with admitting complete induction.